F 415 – Mecânica Geral II – Lista 4

01. P.11.2, Marion:
Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous cone of mass M whose height is h and whose base has a radius R. Choose the x_3-axis along the axis of symmetry of the cone. Choose the origin at the apex of the cone, and calculate the elements of the inertia tensor. Then make a transformation such that the center of mass of the cone becomes the origin, and find the principal moments of inertia.

02. P.11.3, Marion:
Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous ellipsoid of mass M with axes' lengths $2a > 2b > 2c$.

03. P.11.4, Marion:
Consider a thin rod of length l and mass m pivoted about one end. Calculate the moment of inertia. Find the point at which, if all the mass were concentrated, the moment of inertia about the pivot axis would be the same as the real moment of inertia. The distance from this point to the pivot is called the radius of gyration.

P.04. P.11.13, Marion:
A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$m_1 = 3m, \ \ (b, 0, b); \qquad m_2 = 4m, \ \ (b, b, -b); \qquad m_3 = 2m, \ \ (-b, b, 0).$$

Find the Inertia tensor, principal axes, and principal moments of inertia.

05. P.11.14, Marion:
Determine the principal axes and principal moments of inertia of a uniformly solid hemisphere of radius b and mass m about its center of mass.

06. P.11.15, Marion:
If a physical pendulum has the same period of oscillation when pivoted about either of two points of unequal distances from the center of mass, show that the length of the simple pendulum with the same period is equal to the sum of separations of the pivot points from the center of mass. Such a physical pendulum, called Kater's reversible pendulum, at one time provided the most accurate way (to about 1 part in 10^5) to measure the acceleration of gravity. Discuss the advantages of Kater's pendulum over a simple pendulum for such a purpose.

07. P.11.16, Marion: similarity transformation:
Consider the following Inertia tensor:

$$I = \begin{bmatrix} (A+B)/2 & (A-B)/2 & 0 \\ (A-B)/2 & (A+B)/2 & 0 \\ 0 & 0 & C \end{bmatrix}.$$

Perform a rotation of the coordinate system by an angle θ about the x_3-axis. Evaluate the transformed tensor elements, and show that the choice $\theta = \pi/4$ renders the inertia tensor diagonal with elements A, B, and C.

P.08. P.11.17 and P.11.18, Marion:

a) Consider a thin homogeneous plate that lies in the x_1-x_2 plane. Show that the inertia tensor takes the form

$$I = \begin{bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{bmatrix}.$$

b) If the coordinate axes are rotated through an angle θ about the x_3-axis, show that the new inertia tensor is

$$I = \begin{bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A'+B' \end{bmatrix}.$$

where

$$A' = A\cos^2\theta - C\sin 2\theta + B\sin^2\theta, \qquad B' = A\sin^2\theta + C\sin 2\theta + B\cos^2\theta,$$

$$C' = C\cos 2\theta - \frac{1}{2}(B-A)\sin 2\theta,$$

and hence show that the x_1- and x_2-axes become principal axes if the angle of rotation is

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2C}{B-A}\right).$$

09. P.11.22 and P.11.23, Marion:

a) The trace of a tensor is defined as the sum of the diagonal elements:

$$\mathrm{Tr}\, I = \sum_i I_{ii}.$$

Show, by performing a similarity transformation, that the trace is an invariant quantity. In other words, show that

$$\mathrm{Tr}\, I = \mathrm{Tr}\, I',$$

where I is the tensor in one coordinate system and I' is the tensor in a coordinate system rotated with respect to the first system. Verify this result for the different forms of the inertia tensor for a cube given in several examples in the text.

b) Show by the method used in the previous problem that the determinant of the elements of a tensor is an invariant quantity under a similarity transformation. Verify this result also for the case of the cube.

P.10. P.11.27, Marion:
A symmetric body moves without the influence of forces or torques. Let x_3 be the symmetry axis of the body and $\mathbf{L}$ be along x_3'. The angle between $\boldsymbol{\omega}$ and x_3 is α. Let $\boldsymbol{\omega}$ and $\mathbf{L}$ initially be in the x_2-x_3 plane. What is the angular velocity of the symmetry axis about $\mathbf{L}$ in termos of I_1, I_3, ω, and α?

P.11. P.11.29, Marion:
Investigate the motion of the symmetric top discussed in Section 11.11 for the case in which the axis of rotation is vertical (i.e., the x_3'- and x_3-axes coincide). Show that the motion is either stable or unstable depending on whether the quantity $4I_1 Mhg/I_3^2\omega_3^2$ is less than or greater than unity. Sketch the effective potential $V(\theta)$ for the two cases, and point out the features of these curves that determine whether the motion is stable. If the top is set spinning in the stable configuration, what is the effect as friction gradually reduces the value of ω_3? (This is the case of the "sleeping top.")

P.12. P.11.31, Marion:
Consider a thin homogeneous plate with principal momenta of inertia
I_1 along the principal axis x_1,
$I_2 > I_1$ along the principal axis x_2,
$I_3 = I_1 + I_2$ along the principal axis x_3.
Let the origins of the x_i and x_i' systems coincide and be located at the center of mass O of the plate. At time $t = 0$, the plate is set rotating in a force-free manner with an angular velocity Ω about an axis inclined at an angle α from the plane of the plate and perpendicular to the x_2-axis. If $I_1/I_2 = \cos 2\alpha$, show that at time t the angular velocity about the x_2-axis is

$$\omega_2(t) = \Omega \cos\alpha \tanh\left(\Omega t \sin\alpha\right).$$

13. P.11.34, Marion:
Consider a symmetrical rigid body rotating freely about its center of mass. A frictional torque $N_f = -b\omega$ acts to slow down the rotation. Find the component of the angular velocity along the symmetry axis as a function of time.

P.14. P.10.37, Taylor:
A thin, flat, uniform metal triangle lies in the xy plane with its corners at $(1, 0, 0)$, $(0, 1, 0)$, and the origin. Its surface density (mass/area) is $\sigma = 24$. (Distances and masses are measured in unspecified units, and the number 24 was chosen to make the answer come out nicely.)

a) Find the triangle's inertia tensor $\mathbf{I}$.

b) What are its principal moments and the corresponding axes?

15. P.10.2, Symon:
Show that the centrifugal force $\mathbf{F}_c = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a linear function of the position vector $\mathbf{r}$ of the particle, and find an expression for the corresponding tensor in dyadic form, i.e., show that $\mathbf{F}_c = \mathbf{T} \cdot \mathbf{r}$. Write out the matrix of its coefficients.

16. P.10.3, Symon:
For a tensor $\mathbf{T}$, define time derivatives $(d\mathbf{T}/dt)_f$ and $(d\mathbf{T}/dt)_r$ relative to fixed and rotating coordinate systems, as was done in Chapter 7 for derivatives of vectors. Prove that

$$\left(\frac{d\mathbf{T}}{dt}\right)_f = \left(\frac{d\mathbf{T}}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{T} - \mathbf{T} \times \boldsymbol{\omega},$$

where the cross product of a vector with a tensor is defined in the obvious way.

17. P.10.5 and 10.12, Symon:
Transform the tensor $\mathbf{T} = \mathbf{AB} + \mathbf{BA}$, where $\mathbf{A} = 5\hat{x} - 3\hat{y} + 2\hat{z}$ and $\mathbf{B} = 5\hat{y} + 10\hat{z}$, into a coordinate system rotated 45° about the z-axis, using Eq. (10.74). Transform the vectors $\mathbf{A}$ and $\mathbf{B}$, using Eq. (10.71), and show that the results agree. Diagonalize the tensor $\mathbf{T}$, i.e., find its eigenvalues and the corresponding principal axes.

P.18. P.11.4 and P.11.5, Symon:

a) Show that if the only torque on a symmetrical rigid body is about the axis of symmetry, then $(\omega_1^2 + \omega_3^2)$ is constant, where ω_1 and ω_2 are angular velocity components along axes perpendicular to the symmetry axis. If $N_3(t)$ is given, show how to solve for ω_1, ω_2, and ω_3.

b) A symmetrical rigid body ($I_1 = I_2 \neq I_3$) moving freely in space is powered with jet engines symmetrically placed with respect to the body x_3–axis, which supply a constant torque N_3 about the symmetry axis. Find the solution for the angular velocity vector as a function of time relative to the body axes and describe how the angular velocity vector moves relative to the body.

19. P.11.6, Symon:

a) Consider a charged sphere whose mass m and charge e are both distributed in a spherically symmetrical way. That is, the mass and the charge densities are each functions of the radius r (but not necessarily the same function). Show that if this body rotates in a uniform magnetic field $\mathbf{B}$, the torque on it is (Gaussian units)

$$\mathbf{N} = \frac{eg}{2mc}\mathbf{L} \times \mathbf{B},$$

where g is a numerical constant, and $g = 1$ if the mass density is everywhere proportional to the charge density.

b) Write an equation of motion for the body, and show that by introducing a suitably rotating coordinate system, you can eliminate the magnetic torque.

c) Compare this result with Larmor's theorem (Chapter 7). Why is no assumption needed here regarding the strength of the magnetic field?

d) Describe the motion. What points in the body are at rest in the rotating coordinate system?

P.20. P.11.14, Symon:
Discuss the free rotation of a symmetrical rigid body, using the Lagrangian method. Find the angular velocity for uniform precession and the frequency of small nutations about this uniform precession. Describe the motion and show that your results agree with the solutions found in Section 11.2 and in Problem 11.3, Symon.

P.21. P.11.17, Symon (opcional):
A planet consists of a uniform sphere of radius a, mass M, girdled at its equator by a ring of mass m. The planet moves (in a plane) about a star of mass M'. Set up the Lagrangian function, using as coordinates the polar coordinates r and α in the plane of the orbit, and Euler's angles θ, ϕ, and ψ, relative to space axes of which the z-axis is perpendicular to the plane of the orbit, and the x-axis is parallel to the axis from which α is measured. You may assume that $r \gg a$, and use the result of Problem 15, Chapter 6. Find the ignorable coordinates, and show that the period of rotation of the planet is constant.

22. P.11.20, Symon:
Write Lagrangian equations of motion for the rigid body in Problem 11.5, Symon. Carry the solution as far as you can. [Make use of the results of Problem 5 if you wish.] Show that you can obtain a second order differential equation involving θ alone. Can you find any particular solutions, or approximate solutions, of this equation for special cases? Describe the corresponding motions. [Note that this problem, to the extent that it can be solved, gives the motion of the body in space, in contrast to Problem 5, where we found the angular velocity relative to the body.]

23. P.11.21, Symon:
An electron may for some purposes be regarded as a spinning charged sphere like that considered in Problem 11.6, Symon, with g very nearly equal to 2. Show that if g were exactly 2, and the electron spin angular momentum is initially parallel to its linear velocity, then as the electron moves through any magnetic field, its spin angular momentum would always remain parallel to its velocity.

24. P.5.3, Goldstein:
Prove that for a general rigid body motion about a fixed point, the time variation of the kinetic energy T is given by

$$\frac{dT}{dt} = \boldsymbol{\omega} \cdot \mathbf{N}.$$

25. P.5.8, Goldstein:
When the rigid body is not symmetrical, an analytic solution to Euler's equation for the torque-free motion cannot be given hi terms of elementary functions. Show, however, that the conservation of energy and angular momentum can be used to obtain expressions for the body components of $\boldsymbol{\omega}$ in terms of elliptic integrals.

P.26. P.5.9, Goldstein:
Apply Euler's equations of motion to the problem of the heavy symmetrical top, expressing w_i in terms of the Euler angles. Show that the two constants of motion (p_ϕ and p_ψ) can be obtained directly from Euler's equations in this form.

P.27. P.5.10, Goldstein:
Obtain from Euler's equations of motion the condition $Mgh = \dot{\phi}\left(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0\right)$ for the uniform precession of a symmetrical top in a gravitational field, by imposing the requirement that the motion be a uniform precession without nutation.

P.28. P.5.11, Goldstein:
Show that the magnitude of the angular momentum for a heavy symmetrical top can be expressed as a function of θ and the constants of motion only. Prove that as a result the angular momentum vector precesses uniformly only when there is uniform precession of the symmetry axis.

29. P.5.16, Goldstein:
Three equal mass points are located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a.a)$. Find the principal moments of inertia about the origin and a set of principal axes.