F 415 – Mecânica Geral II – Lista 5

P.01. P.12.1, Marion: Two springs and two particles:
Reconsider the problem of two coupled oscillators discussed in Section 12.2 in the event that the three springs all have different force constants. Find the two characteristic frequencies, and compare the magnitudes with the natural frequencies of the two oscillators in the absence of coupling.

02. P.12.4, Marion: Two springs and two particles:
Refer to the problem of the two coupled oscillators discussed in Section 12.2. Show that the total energy of the system is constant. (Calculate the kinetic energy of each of the particles and the potential energy stored in each of the three springs, and sum the results.) Notice that the kinetic and potential energy terms that have κ_{12} as a coefficient depend on C_1 and ω_1 but not on C_2 or ω_2. Why is such a result to be expected?

03. P.12.5, Marion: Two springs and two particles:
Find the normal coordinates for the problem discussed in Section 12.2 and in Example 12.1 if the two masses are different, $m_1 \neq m_2$. You may again assume all the κ are equal.

P.04. P.12.7, Marion: Two springs, two particles, and gravity:
A particle of mass m is attached to a rigid support by a spring with force constant K. At equilibrium, the spring hangs vertically downward. To this mass-spring combination is attached an identical oscillator, the spring of the latter being connected to the mass of the former. Calculate the characteristic frequencies for one-dimensional vertical oscillations, and compare with the frequencies when one or the other of the particles is held fixed while the other oscillates. Describe the normal modes of motion for the system.

05. P.12.4, Symon: Two springs, two particles, and gravity:
A mass m is hung from a fixed support by a spring of constant k whose relaxed length is $l = 2mg/k$. A second equal mass is hung from the first mass by an identical spring. Find the six normal coordinates and the corresponding frequencies for small vibrations of this system from its equilibrium position. Each spring exerts a force only along the line joining its two ends, but may pivot freely in any direction at its ends. Neglect the mass of the springs.
Obs.: Consider that the motion is in the plane.

06. P.12.8, Marion: Double pendulum:
A simple pendulum consists of a bob of mass m suspended by an inextensible (and massless) string of length l From the bob of this pendulum is suspended a second, identical pendulum. Consider the case of small oscillations (so that $\sin\theta \approx \theta$), and calculate the characteristic frequencies. Describe also the normal modes of the system (refer to Problem 7-7).

P.07. P.6.4 Goldstein: Double pendulum:
Obtain the normal modes of vibration for the double pendulum shown in Fig. 12.E, Marion assuming equal lengths, but not equal masses, i.e., $L_1 = L_2$ and $m_1 \neq m_2$. Show that when the lower mass is small compared to the upper one, the two normal frequencies are almost equal. If the pendula are set in motion by pulling the upper mass slightly away from the vertical and then releasing it, show that the subsequent motion is such that at regular intervals one pendulum is at rest while the other has its maximum amplitude. This is the familiar phenomenon of "beats".

P.08. P.12.10, Marion: Two oscillators + dissipation + external force:
Consider two identical, coupled oscillators (as in Figure 12-1). Let each of the oscillators be damped, and let each have the same damping parameter β. A force $F_0 \cos \omega t$ is applied to m_1. Write down the pair of coupled differential equations that describe the motion. Obtain the solution by expressing the differential equations in terms of the normal coordinates given by Equation 12.11 and by comparing these equations with Equation 3.53. Show that the normal coordinates η_1 and η_2 exhibit resonance peaks at the characteristic frequencies ω_1 and ω_2, respectively.

09. P.12.13, Marion: LC circuit:
Find the characteristic frequencies of the coupled circuits of Figure 12-C.

P.10. P.12.14, Marion: LC circuit:
Discuss the normal modes of the system shown in Figure 12-D.

11. P.12.16, Marion: Thin hoop and 1 particle:
A thin hoop of radius R and mass M oscillates in its own plane hanging from a single fixed point. Attached to the hoop is a small mass M constrained to move (in a frictionless manner) along the hoop. Consider only small oscillations, and show that the eigenfrequencies are

$$\omega_1 = \sqrt{2g/R}, \qquad \omega_1 = \sqrt{g/(2R)},$$

Find the two sets of initial conditions that allow the system to oscillate in its normal modes. Describe the physical situation for each mode.

P.12. P.12.17, Marion: Four springs and three particles:
Find the eigenfrequencies and describe the normal modes for a system such as the one discussed in Section 12.2 but with three equal masses m and four springs (all with equal force constants) with the system fixed at the ends.

13. P.12.15, Symon:
A triple pendulum is formed by suspending a mass M by a string of length l from a fixed support. A mass m is hung from M by a string of length l and from this second mass a third mass m is hung by a third string of length l. The masses swing in a single vertical plane. Set up the equations for small vibrations of the system, using as coordinates the angles θ_1, θ_2, and θ_3 made by each string with the vertical. Show that if $M \gg m$, the normal coordinates can be found if terms of order $(m/M)^{1/2}$ are neglected. Find the approximate normal frequencies to order m/M.
Hint: Transform matrix $\hat{K}$ to a constant tensor and diagonalize matrix $\hat{M}$.

P.14. P.6.8 Goldstein: triatomic molecule:
The equilibrium configuration of a molecule is represented by three atoms of equal mass at the vertices of a $45°$ right triangle connected by springs of equal force constants. Obtain the secular determinant for the modes of vibration in the plane and show by rearrangement of the columns that the secular equation has a triple root $\omega = 0$. Reduce the determinant to one of third rank and obtain the nonvanishing frequencies of free vibration.

15. P.12.18, Marion:
A mass M moves horizontally along a smooth rail. A pendulum is hung from M with a weightless rod and mass m at its end. Find the eigenfrequencies and describe the normal modes.

P.16. P.12.5, Symon (opcional):
An ion of mass m and charge q is held by a linear attractive force $F = -kr$ to a point A, where r is the distance from the ion to the point A. An identical ion is similarly bound to a second point B a distance d from B. The two ions move (in three-dimensional space) under the action of these forces and their mutual electrostatic repulsion. Find the normal modes of vibration, and write down the most general solution for small vibrations about the equilibrium point.
Obs.: consider that the ions move only along the x-direction.

17. P.6.13 Goldstein:
Two particles (mass m) are connected to each other and to fixed points by three equal springs of force constant k (see Fig. 12.1, Marion). The equilibrium length of each spring is a. Each mass point has a positive charge $+q$ and they repel each other according to Coulomb law. Set up the secular equation for the eigenfrequencies.

18. P.12.24, Marion: N particles:
Show that the equations of motion for longitudinal vibrations of a loaded string are of exactly the same form as the equations for transverse motion (Equation 12.131), except that the factor τ/d must be replaced by K, the force constant of the string.

P.19. P.13.1, Marion: String and initial conditions:
Discuss the motion of a continuous string when the initial conditions are $\dot{q}(x,0) = 0$ and $q(x,0) = A\sin(3\pi x/L)$. Resolve the solution into normal modes.

20. P.13.2, Marion: String and initial conditions:
Rework the problem in Example 13.1 in the event that the plucked point is a distance $L/3$ from one end. Comment on the nature of the allowed modes.

21. P.13.4, Marion: String and initial conditions:
Discuss the motion of a string when the initial conditions are $q(x,0) = 4x(L-x)/L^2$, $\dot{q}(x,0) = 0$. Find the characteristic frequencies and calculate the amplitude of the n-th mode.

P.22. P.13.6, Marion: String and initial conditions:
A string is set into motion by being struck at a point $L/4$ from one end by a triangular hammer. The initial velocity is greatest at $x = L/4$ and decreases linearly to zero at $x = 0$ and $x = L/2$. The region $L/2 \leq x \leq L$ is initially undisturbed. Determine the subsequent motion of the string. Why are the fourth, eighth, and related harmonics absent? How many decibels down from the fundamental are the second and third harmonics?

23. P.13.11, Marion: String and external force:
When a particular driving force is applied to a string, it is observed that the string vibration is purely of the $n-$th harmonic. Find the driving force.

24. P.13.13, Marion:
Consider the simplified wave function $\Psi(x,t) = A\exp\left(i(\omega t - kx)\right)$. Assume that ω and v are complex quantities and that k is real: $\omega = \alpha + i\beta$ and $v = u + i\omega$. Show that the wave is damped in time. Use the fact that $k^2 = \omega^2/v^2$ to obtain expressions for α and β in terms of u and ω. Find the phase velocity for this case.

25. P.13.15, Marion: Beats:
Consider the superposition of two infinitely long wave trains with almost the same frequencies but with different amplitudes. Show that the phenomenon of beats occurs but that the waves never beat to zero amplitude.

P.26. P.13.17, Marion: Strings and particles (mass $m' \neq m''$):
Treat the problem of wave propagation along a string loaded with particles of two different masses, m' and m", which alternate in placement; that is, $m_j = m'$ and m'' respectively for j even and odd. Show that the $\omega - k$ curve has two branches in this case and show that there is attenuation for frequencies between the branches as well as for frequencies above the upper branch.

27. P.13.18, Marion:
Sketch the phase velocity $V(k)$ and the group velocity $U(k)$ for the propagation of waves along a loaded string in the range of wave numbers $0 \leq k \leq \pi/d$. Show that $U(\pi/d) = 0$, whereas $V(\pi/d)$ does not vanish. What is the interpretation of this result in terms of the behavior of the waves?

P.28. P.13.19, Marion: Transmission and reflection:
Consider an infinitely long continuous string with linear mass density ρ_1 for $x < 0$ and for $x > L$, but density $\rho_2 > \rho_1$ for $0 < x < L$. If a wave train oscillating with an angular frequency ω is incident from the left on the high-density section of the string, find the reflected and transmitted intensities for the various portions of the string. Find a value of L that allows a maximum transmission through the high-density section. Discuss briefly the relationship of this problem to the application of nonreflective coatings to optical lenses.

29. P.13.20, Marion: Transmission and reflection:
Consider an infinitely long continuous string with tension τ. A mass M is attached to the string at $x = 0$. If a wave train with velocity ω/k is incident from the left, show that reflection and transmission occur at $x = 0$ and that the coefficients R and T are given by $R = \sin^2\theta$ and $T = \cos^2\theta$, where $\tan\theta = M\omega^2/(2k\tau)$. Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$. What are the pheise changes for the reflected and transmitted waves?

30. P.13.21, Marion: Wave packet:

P.31. P.13.22, Marion: Gaussian wave packet:

32. P.8.1, Symon:
A stretched string of length l is terminated at the end $x = l$ by a ring of negligible mass which slides without friction on a vertical rod.

a) Show that the boundary condition at this end of the string is $(\partial\Psi(x,t)/\partial x)_{x=l} = 0$.
b) If the end $x = 0$ is tied, find the normal modes of vibration.

P.33. P.8.6, Symon:
A force of linear density $f(x,t) = f_0 \sin(n\pi x/l) \cos\omega t$, where n is an integer, is applied along a stretched string of length l.

a) Find the steady-state motion of the string.
Hint: Assume a similar time and space dependence for $\Psi(x,t)$ and substitute in the equation of motion.
b) Indicate how one might solve the more general problem of a harmonic applied force $f(x,t) = f_0(x)\cos\omega t$, where $f_0(x)$ is any function which vanishes at the ends.