F 415 – Mecânica Geral II – Lista 6

01. P.8.8, Goldstein:
Show that the modified Hamilton's principle, in the form

$$\delta \int_{t_1}^{t_2} \left(-\dot{p}_i q_i - H(q, p, t)\right) dt = 0$$

leads to Hamilton's equations of motion.

P.02. P.7.27, Marion:
A massless spring of length b and spring constant k connects two particles of masses m_1 and m_2. The system rests on a smooth table and may oscillate and rotate.

a) Determine Lagrange's equations of motion.

b) What are the generalized momenta associated with any cyclic coordinates?

c) Determine Hamilton's equations of motion.

P.03. P.7.29, Marion:
A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. The pendulum's point of support rises vertically with constant acceleration a.

a) Use the Lagrangian method to find the equations of motion.

b) Determine the Hamiltonian and Hamilton's equations of motion.

c) What is the period of small oscillations?

P.04. P.8.13, Goldstein:
Formulate the double-pendulum problem illustrated by Fig. 1.4, in terms of the Hamiltonian and Hamilton's equations of motion. It is suggested that you find the Hamiltonian both directly from L and by Eq. (8.27). Consider the limit of small oscillations and assume that the masses $m_1 = m_2 = m$ and the length of the (massless and inextensible) strings $l_1 = l_2 = l$ (see P.12.8, Marion).

Hint: Write the canonical momenta p_1 and p_2 in terms of θ_1 and θ_2 as $\hat{p} = \hat{A}\dot{\hat{\theta}}$, where the vectors $\hat{p}^T = (p_1\ p_2)$, $\dot{\hat{\theta}}^T = \left(\dot{\theta}_1\ \dot{\theta}_2\right)$, and $\hat{A}$ is a 2×2 matrix.

05. P.8.19, Goldstein:
The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola $z = ax^2$ in the vertical plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.

P.06. P.8.20, Goldstein:
Obtain Hamilton's equations of motion for a plane pendulum of length l with mass point m whose radius of suspension rotates uniformly on the circumference of a vertical circle of radius a. Describe physically the nature of the canonical momentum and the Hamiltonian.

P.07. P.8.24, Goldstein (opcional):
A uniform cylinder of radius a and density ρ is mounted so as to rotate freely around a vertical axis. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction. Suppose a particle starts at rest at the top of the cylinder and slides down under the influence of gravity. Using any set of coordinates, arrive at a Hamiltonian for the combined system of particle and cylinder, and solve for the motion of the system.

08. P.7.24, Marion:
Consider a simple plane pendulum consisting of a mass m attached to a string of length I. After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{ cte.}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

P.09. P.8.27, Goldstein:

a) The Lagrangian for a system of one degree of freedom can be written as

$$L = \frac{1}{2}m\left(\dot{q}^2 \sin^2 \omega t + \dot{q}q\omega \sin 2\omega t + q^2\omega^2\right).$$

What is the corresponding Hamilionian? Is it conserved?

b) Introduce a new coordinate defined by

$$Q = q \sin \omega t.$$

Find the Lagrangian in terms of the new coordinate and the corresponding Hamiltonian. Is H conserved?

10. P.8.35, Goldstein:
Consider a Lagrangian of the form

$$L = \frac{1}{2}m\left(\dot{x}^2 - \omega^2 x^2\right)e^{\gamma t},$$

where the particle of mass m moves in one direction. Assume all constants are positive.

a) Find the equations of motion.

b) Interpret the equations by giving a physical interpretation of the forces acting on the particle.

c) Find the canonical momentum and construct the Hamiltonian. Is this Hamiltonian a constant of the motion?

d) If initially $x(0) = 0$ and $dx/dt = 0$, what is $x(t)$ as t approaches large values?

11. P.7.30, Marion: Poisson brackets.

12. P.9.2, Goldstein:
Show that the transformation for a system of one degree of freedom,

$$Q = q\cos\alpha - p\sin\alpha \qquad \text{and} \qquad P = q\sin\alpha + p\cos\alpha,$$

satisfies the symplectic condition for any value of the parameter α. Find a generating function for the transformation. What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both of these cases.

P.13. P.9.6, Goldstein:
The transformation equations between two sets of coordinates are

$$Q = \ln\left(1 + q^{1/2}\cos p\right) \qquad \text{and} \qquad P = 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p$$

a) Show directly from these transformation equations that Q and P are canonical variables if q and p are.

b) Show that the function that generates this transformation is

$$F_3 = -\left(e^Q - 1\right)^2 \tan p.$$

14. P.9.7, Goldstein:

a) If each of the four types of generating functions exist for a given canonical transformation, use the Legendre transformation to derive relations between them.

b) Find a generating function of the F_4 type for the identify transformation and of the F_3, type for the exchange transformation.

c) For an orthogonal point transformation of q in a system of n degrees of freedom, show that the new momenta are likewise given by the orthogonal transformation of an n-dimensional vector whose components are the old momenta plus a gradient in configuration space.

15. P.9.8, Goldstein:
Prove directly that the transformation

$$Q_1 = q_1, \qquad P_1 = p_1 - 2p_2,$$

$$Q_2 = p_2, \qquad P_2 = -2q_1 - q_2$$

is canonical and find a generating function.

16. P.9.10, Goldstein:
Find under what conditions

$$Q = \alpha \frac{p}{x} \qquad \text{and} \qquad P = \beta x^2,$$

where α and β are constants, represents a canonical transformation for a system of one degree of freedom, and obtain a suitable generating function. Apply the transformation to the solution of the linear harmonic oscillator.

P.17. P.9.15, Goldstein:

a) Using the fundamental Poisson brackets find the values of α and β for which the equations

$$Q = q^\alpha \cos \beta p \qquad \text{and} \qquad P = q^\alpha \sin \beta p$$

represent a canonical transformation.

b) For what values of α and β do these equations represent an extended canonical transformation? Find a generating function of the F_3 form for the transformation.

c) On the basis of part (b) can the transformation equations be modified so that they describe a canonical transformation for all values of β?

P.18. P.9.21, Goldstein (opcional):

a) For a one-dimensional system with the Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{2q^2},$$

show that there is a constant of the motion

$$D = \frac{1}{2}pq - Ht.$$

b) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = p^n - ar^{-n}, \qquad p = |\mathbf{p}|,$$

where $\mathbf{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{1}{n}\mathbf{p}\cdot\mathbf{r} - Ht.$$

c) The transformation $Q = \lambda q$, $p = \lambda P$ is obviously canonical. However, the same transformation with t time dilatation, $Q = \lambda q$, $p = \lambda P$,$t' = \lambda^2 t$, is not. Show that, however, the equations of motion for q and p for the Hamiltonian in part (a) are invariant under this transformation. The constant of the motion D is said to be associated with this invariance.

19. P.9.22, Goldstein:
For the point transformation in a system of two degrees of freedom,

$$Q_1 = q_1^2, \qquad Q_2 = q_1 + q_2,$$

find the most general transformation equations for P_1 and P_2 consistent with the overall transformation being canonical. Show that with a particular choice for P_1 and P_2 the Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

can be transformed to one in which both Q_1 and Q_2 are ignorable. By this means solve the problem and obtain expressions for q_1, q_2, p_1, and p_2 as functions of time and their initial values.

P.20. P.9.24, Goldstein:

a) Show that the transformation

$$Q = p + iaq, \qquad P = \frac{1}{2ia}\,(p - iaq)$$

is canonical and find a generating function.

b) Use the transformation to solve the linear harmonic oscillator problem.

21. P.9.25, Goldstein:

a) The Hamiltonian for a system has the form

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right).$$

Find the equation of motion for q.

b) Find a canonical transformation that reduces H to the form of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found in part (a) is satisfied.

P.22. P.9.31, Goldstein:
Show by the use of Poisson brackets that for a one-dimensional harmonic oscillator there is a constant of the motion u defined as

$$u(q, p, t) = \ln(p + im\omega q) - i\omega t, \qquad \omega = \sqrt{k/m}.$$

What is the physical significance of this constant of the motion?

23. P.9.41, Goldstein:
We start with a time independent Hamiltonian $H_0(q, p)$ and impose an external oscillating field making the Hamiltonian

$$H = H_0(q, p) - \epsilon \sin \omega t,$$

where ϵ and ω are given constants.

a) How are the canonical equations modified?

b) Find a canonical transformation that restores the canonical form of the equations of motion and determine the "new" Hamiltonian.

c) Give a possible physical interpretation of the imposed field.