Ex. 18.8.3 Problema 5.23 Griffiths

Dado $\vec{A} = k\hat{\phi}$, k constante, obter a densidade de corrente $\vec{J}$ que o geraria

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat\rho & \rho\hat\phi & \hat z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ 0 & \rho k & 0 \end{vmatrix} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(k\rho)\,\hat z$$

$$\Rightarrow \vec{B} = (k/\rho)\,\hat z$$

(Notem que $\nabla \cdot \vec{A} = \partial k/\partial\phi = 0$)

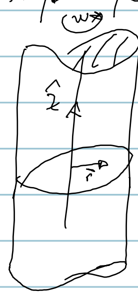
Agora $\nabla \times \vec{B} = \mu_0 \vec{J}$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0}\frac{k}{\rho}\begin{vmatrix} \hat\rho & \rho\hat\phi & \hat z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ 0 & 0 & 1/\rho \end{vmatrix} = -\frac{k}{\mu_0}\frac{\cancel{\rho}}{\cancel{\rho}}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right)\hat\phi$$

$$\boxed{\vec{J} = \frac{k}{\mu_0 \rho^2}\,\hat\phi}$$

Obs.: Um cilindro girando com $\omega = \omega\hat z$ e densidade de carga $\bar\rho(\rho)$ tem $\vec J$

$$\vec{J} = (\vec\omega \times \vec\rho)\,\bar\rho(\rho)$$



$$\vec\omega \times \vec\rho = \omega\rho\,\hat\varphi$$

$$\vec{J} = \omega\rho\bar\rho(\rho)\,\hat\varphi$$

Se $\omega\rho\bar\rho(\rho) = \cancel{k}^{\,\omega}/\rho^2$, basta que $\bar\rho(\rho) = 1/\rho^3$ para termos o campo $\vec B$ em $\hat z$

Ex. 18.8.4 Prob. 5.24 Griffiths

$$\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B}) \qquad \vec{\nabla}\cdot\vec{A} = 0 \ ?$$

$\vec{B}$ é uniforme $\quad \vec{\nabla}\times\vec{A} = \vec{B}$?

$$\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = (\vec{\nabla}\times\vec{F})\cdot\vec{G} - (\vec{\nabla}\times\vec{G})\cdot\vec{F}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = -\frac{1}{2}\Big[\underbrace{(\vec{\nabla}\times\vec{r})}_{=0}\cdot\vec{B} - \underbrace{(\vec{\nabla}\times\vec{B})}_{=0}\cdot\vec{r}\Big] = 0 \quad \checkmark$$

$$\vec{\nabla}\times(\vec{F}\times\vec{G}) = (\vec{\nabla}\cdot\vec{G})\vec{F} - (\vec{\nabla}\cdot\vec{F})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{F}\cdot\vec{\nabla})\vec{G}$$

$$-2\,\vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{r}\times\vec{B}) =$$

$$= \underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}\vec{r} - \underbrace{(\vec{\nabla}\cdot\vec{r})}_{3}\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{r} - \underbrace{(\vec{r}\cdot\vec{\nabla})\vec{B}}_{=0}$$

$$-2\,\vec{\nabla}\times\vec{A} = -3\vec{B} + \sum_i B_i \frac{\partial}{\partial x_i}\sum_j x_j\hat{x}_j =$$

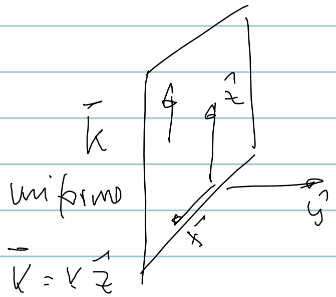
$$= -3\vec{B} + \sum_{ij} B_i \frac{\partial x_j}{\partial x_i}\hat{x}_j =$$

$$= -3\vec{B} + \sum_{ij} B_i\,\delta_{ij}\,\hat{x}_j = -3\vec{B} + \vec{B} = -2\vec{B}$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \vec{B} \quad \checkmark$$

$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ com λ sendo solução da eq. de Laplace também satisfaz.

Ex. 18.8.5 Prob. 5.26 Griffiths



~~Caso~~ $\bar A$ p/ um plano infinito c/ K uni

Sabemos que o campo é uniforme

$$\bar B = -\frac{\mu_0 K}{2}\hat x \qquad (y>0)$$

$$\bar A = -\frac{1}{2}\left(\vec r \times \bar B\right)$$ conforme sugere o exemplo 18.8.4

$$\bar A = -\frac{1}{2}\left(-\frac{\mu_0 K}{2}\right)\vec r \times \hat x = \frac{K\mu_0}{4}\left(z\hat y - y\hat z\right)$$

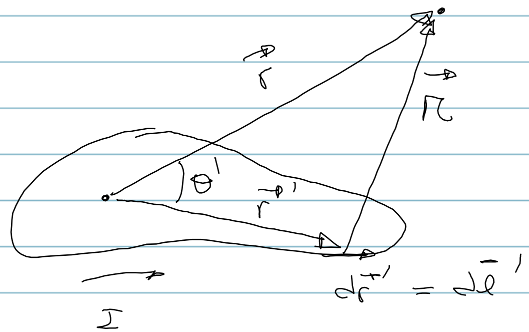
$$\boxed{\bar A = \frac{K}{4}\mu_0\left(z\hat y - y\hat z\right)}$$

Notem que $\bar\nabla \times \bar A = \frac{K}{4}\mu_0 \begin{vmatrix} \hat x & -\hat y & \hat z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & z & -y \end{vmatrix}$

$$= \frac{K}{4}\mu_0\left(-\frac{\partial y}{\partial y}\hat x - \frac{\partial z}{\partial z}\hat x\right) = -\frac{K\mu_0}{2}\hat x$$

18.9 Expansão multipolar

Consideremos um circuito fechado qualquer por onde passa uma corrente estacionária I



$\cos\theta' = \hat{r} \cdot \hat{r}'$

Queremos calcular o vetor potencial em $\vec{r}$

O módulo de $\vec{\mathscr{r}}$ é

$$\mathscr{r} = \left(r^2 + r'^2 - 2rr'\cos\theta'\right)^{1/2}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\theta')}_{\text{polinômio de Legendre}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\underbrace{\oint \frac{1}{\mathscr{r}}\, d\vec{l}'}_{\text{integral em } \vec{r}'} = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\, d\vec{l}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\Bigg[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\substack{n=0 \\ \text{monopolo}}} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}'}_{\substack{n=1 \\ \text{dipolo}}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\vec{l}'}_{\substack{n=2 \\ \text{quadrupolo}}} + \dots$$

Monopolo : $\vec{A}_M = \frac{\mu_0 I}{4\pi r} \underbrace{\oint d\vec{l}'}_{=0} = 0$

Não há termo de monopolo.

Dipolo : $\vec{A}_d(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\vec{l}' =$

$$= \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}'$$

Simplifiquemos :

Chamemos o vetor área $\vec{a} = \int_S d\vec{a}$ → se for plano S $|\vec{a}| = $ área

Seja um vetor $\vec{c}$ constante e um vetor $\vec{v} = \vec{c}\,T$ onde T é uma função escalar.

Apliquemos o teorema de Stokes em $\vec{v}$

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l} = \left(\oint T \, d\vec{l}\right) \cdot \vec{c} \quad (I)$$

Também

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \int_S \vec{\nabla} \times (\vec{c}\,T) \cdot d\vec{a} = \int_S (\vec{\nabla} T \times \vec{c}) \cdot d\vec{a}$$

$$= \int_S (d\vec{a} \times \vec{\nabla} T) \cdot \vec{c} = \left(\int d\vec{a} \times \vec{\nabla} T\right) \cdot \vec{c} \quad (II)$$

Isto vale para qualquer vetor $\vec{c}$ ⇒ I e II leva a

$$\oint T \, d\vec{l} = -\int_S \vec{\nabla} T \times d\vec{a}$$

Suponha que $T = \vec{c} \cdot \vec{r}$; ($\vec{c}$ é constante)

$$\vec{\nabla}(\vec{c}\cdot\vec{r}) = (\vec{c}\cdot\vec{\nabla})\vec{r} + \underbrace{\vec{c}\times(\vec{\nabla}\times\vec{r})}_{=0} + \text{termos}$$

nulos por ter derivadas de $\vec{c}$

$$\Rightarrow \vec{\nabla}(\vec{c}\cdot\vec{r}) = \sum_{ij} c_i \frac{\partial}{\partial x_i} x_j \hat{x}_j$$

$$= \sum_{ij} c_i \delta_{ij} \hat{x}_j = \sum_{i} c_i \hat{x}_i = \vec{c}$$

$\Rightarrow$ $\cancel{\oint T \, d\vec{l} A}$

$$\Rightarrow \oint (\vec{c}\cdot\vec{r})\, d\vec{l} = -\int_S \vec{c}\times d\vec{a} = -\vec{c}\times \underbrace{\int_S d\vec{a}}_{\vec{c}\times\vec{a}}$$

$\vec{a} \to$ vetor área

Voltando à expressão do dipolo:

$$\vec{A}_d(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')\, d\vec{l}' = \frac{\mu_0}{4\pi}\frac{I}{r^2}\left(-\hat{r}\times\int_S d\vec{a}'\right)$$

constante

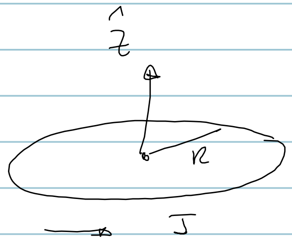
$$\vec{A}_d(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\left(I\int_S d\vec{a}'\right)\times\hat{r}}{r^2}$$

Chamando o momento de dipolo:

$$\vec{m} \equiv I\int_S d\vec{a} \qquad \Big(\text{No caso plano: } \vec{m} = I(\text{área})\hat{n}$$

temos: $$\boxed{\vec{A}_d(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}}$$

Ex. 18.9.1 $\vec{m}$ de uma espira circular



$$\vec{m} = I\pi R^2\hat{z}$$

$$\vec{A}_d(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^2}\,\hat{z}\times\vec{r}$$

$$\vec{A}_d(\vec{r}) = \frac{\mu_0 I R^2}{4r^2}\,\hat{z}\times\vec{r}$$

O campo magnético do dipolo fica:

$$\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\left(\frac{\vec{m}\times\vec{r}}{r^3}\right) =$$

$$= \frac{\mu_0}{4\pi}\left[\underbrace{\nabla\left(\frac{1}{r^3}\right)\times(\vec{m}\times\vec{r})}_{I} + \underbrace{\frac{1}{r^3}\vec{\nabla}\times(\vec{m}\times\vec{r})}_{II}\right]$$

$$I: \; -\frac{3}{r^5}\vec{r}\times(\vec{m}\times\vec{r}) = -\frac{3}{r^5}\left[\vec{m}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{m}\cdot\vec{r})\right]$$

$$= \frac{3(\vec{m}\cdot\hat{r})\hat{r} - 3\vec{m}}{r^3}$$

$$\text{II}: \frac{1}{r^3}\vec{\nabla}\times(\vec{m}\times\vec{r}) = \frac{1}{r^3}\Big[(\vec{\nabla}\cdot\vec{r})\,\vec{m} - \cancel{(\vec{\nabla}\cdot\vec{m})\,\vec{r}}$$

$$\cancel{+(\vec{r}\cdot\vec{\nabla})\,\vec{m}} + \cancel{(\vec{r}\cdot\vec{\nabla})\,\vec{m}} - (\vec{m}\cdot\vec{\nabla})\,\vec{r}\Big]$$

$$= \frac{1}{r^3}\left[3\vec{m} - \vec{m}\right] = \frac{2\vec{m}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[2\vec{m} - 3\vec{m} + 3(\vec{m}\cdot\hat{r})\,\hat{r}\right]$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}\right]}$$

Ex. 18.9.2 campo da espira

$$\vec{m} = I\pi R^2\,\hat{z}$$

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi r^3} I\pi R^2\left[3(\hat{z}\cdot\hat{r})\,\hat{r} - \hat{z}\right]$$

$$\vec{B}_{dip} = \frac{\mu_0 I\cancel{\pi} R^2}{4\cancel{\pi} r^3}\left[3\cos\theta\,\hat{r} - \hat{z}\right]$$

No eixo $\hat{z}$: $\hat{r} = \hat{z}$ e $\cos\theta = 1 \Rightarrow \vec{B} = \dfrac{\mu_0 I R^2\, 2\hat{z}}{4 z^3}$

$r = z$

$$\bar{B}_{dip} = \frac{\mu_0 I}{2} \frac{R^2}{z^3} \hat{z}$$

Do livro Δ p. 151 obtivemos o $\bar{B}$ exato em $\hat{z}$:

$$\bar{B} = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \hat{z}$$

$$\lim_{z \gg R} \bar{B} \to \frac{\mu_0 I R^2}{2 z^3} \hat{z} = \bar{B}_{dip}.$$

Sugestão de problema: 5.35 e 5.36