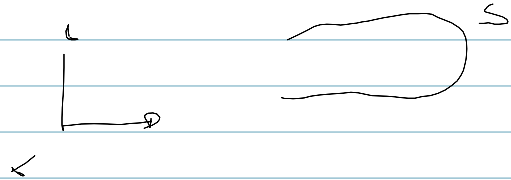
Ex 5.5.4 Fluxo do vetor $\vec{r}$ numa superfície não incluindo a origem com área S e volume V

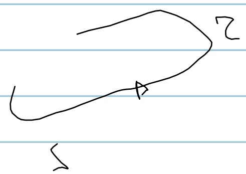


$$\oint_S \vec{r} \cdot d\vec{A} = \int_V (\vec{\nabla} \cdot \vec{r})\, dV = 3\int_V dV = 3V$$

(só depende do volume)

Ex. 5.5.5

Dada uma linha e obter $\oint \vec{r} \cdot d\vec{\ell}$



$$\oint \vec{r} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{r}) \cdot d\vec{A} = 0,$$

pois $\vec{\nabla} \times \vec{r} = 0$

5.6 Laplaciano de um vetor

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \overbrace{\vec{\nabla} (\vec{\nabla}\, \vec{A})}^{BAC} - \overbrace{\vec{A} (\vec{\nabla}\, \vec{\nabla})}^{CAB}$$

Notem que para fazer sentido os operadores devem ser definidos

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\cdot\vec{\nabla}(\vec{A})$$

$$= \vec{\nabla}(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

(Isto é uma maneira utilitária de demonstrar a relação vetorial - operador)

Surge o Laplaciano de um vetor que nada mais é que

$$\nabla^2\vec{A} = \cancel{\nabla^2 A_x \hat{x}}\ (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$$

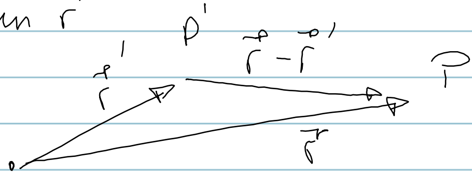
(isto só é válido em coordenadas cartesianas)

Ex. 5.6.1 $\nabla^2\vec{r} = \cancel{\partial x}(\nabla^2 x)\hat{x} + (\nabla^2 y)\hat{y} + (\nabla^2 z\,\hat{z})$

$$\nabla^2 x = \frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 x}{\partial z^2} = 0$$

$$\nabla^2\vec{r} = 0$$

Ex. 5.6.2 É comum termos o problema descrito em termos de um vetor que vai de um ponto P' em $\vec{r}'$ a um ponto P em $\vec{r}$



Seja uma função escalar que depende de $|\vec{r}-\vec{r}'|$ (a distância entre P e P').
Por exemplo

$$\varphi = \frac{1}{|\vec{r}-\vec{r}'|}$$

Se quisermos saber o gradiente de φ, temos que especificar se movemos P ou P'.

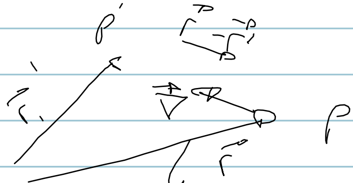
Se for P gradiente $\rightarrow \vec{\nabla}$

$$\nabla \varphi = \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}$$

$$\vec{\nabla}\varphi = -\frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\vec{\nabla}\varphi = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

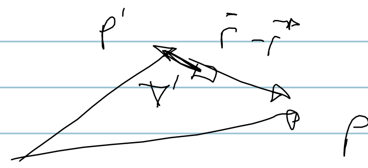


aponta no sentido de aproximar P a P'

Se for P' gradiente $\rightarrow \vec{\nabla}'$

$$\vec{\nabla}'\varphi = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

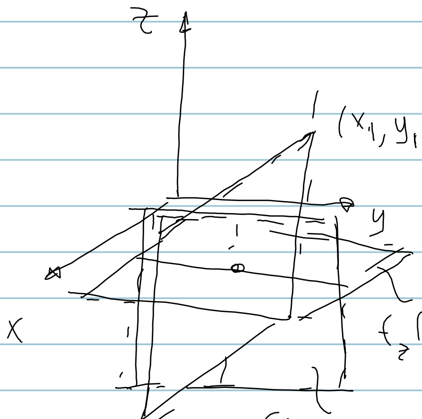
$\left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$



6 - Coordenadas Curvilíneas

(Não vamos entrar em muito detalhe)

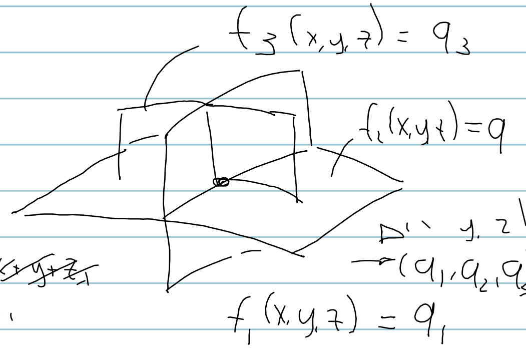
Assim como nas coordenadas cartezianas um ~~plano~~ ponto é determinado pela intersecção de três planos, poderíamos ter superfícies ortogonais ~~para~~ cuja intersecção determina um ponto.



$f_z(x,y,z) = \cancel{x+y+z_1}$ $z = z_1$

$f_x(x,y,z) = \cancel{x_1+y_1+z}$ $x_1 = x$

$f_y(x,y,z) = \cancel{x_1+y_1+z}$ $y_1 = y$



As novas coordenadas são q_1, q_2 e q_3

Uma questão importante é a métrica destas superfícies e as coordenadas.

Nas coordenadas cartezianas um elemento de comprimento perpendicular a cada superfície é

$$dl_x = dx$$

$$dl_y = dy \quad ; \quad dl_z = dz$$

Nas coordenadas q, no entanto:

$$dl_1 = h_1(q_1, q_2, q_3)\, dq_1$$

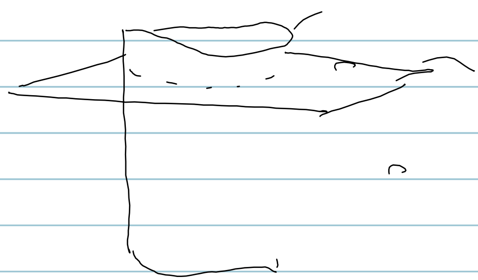
$$dl_2 = h_2(q_1, q_2, q_3)\, dq_2$$

$$dl_3 = h_3(q_1, q_2, q_3)\, dq_3$$

Da mesma forma o elemento de volume fica então $dl_1 dl_2 dl_3 = h_1 h_2 h_3\, dq_1 dq_2 dq_3$

~~Ex~~ 6.1 Coordenadas cilíndricas

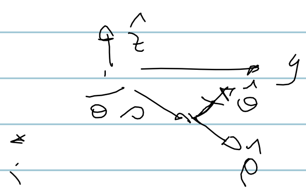
cilindros de raio $\rho = (x^2+y^2)^{1/2}$



planos $z = z$

~~planos~~ $\theta = \mathrm{arctg}\left(\frac{y}{x}\right)$ (meio plano)

Note que: $\rho \to q_1$, $\theta \to q_2$, $z \to q_3$

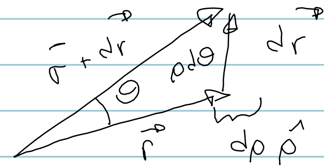


$$dl_1 = d\rho$$
$$dl_2 = \rho\, d\theta$$
$$dl_3 = dz$$

$$\Rightarrow h_1 = 1;\quad h_2 = \rho;\quad h_3 = 1$$

$$\boxed{dV = \rho\, d\rho\, d\theta\, dz}$$

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$



$$d(\rho\hat{\rho}) = d\rho\,\hat{\rho} + \rho\, d\theta\,\hat{\theta}$$

(Mantendo z constante

$$\boxed{d\vec{r} = d\rho\,\hat{\rho} + \rho\, d\theta\,\hat{\theta} + dz\,\hat{z}}$$

$$\left(d\hat{\rho} = d\theta\,\hat{\theta} \right)$$

Gradiente.

$$\vec{\nabla} f = \frac{\partial f}{\partial l_1}\hat{\rho} + \frac{\partial f}{l_2}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

$$\frac{\partial f}{\partial l_1} = \frac{\partial f}{\partial \rho}\frac{d\rho}{dl_1} = \frac{\partial f}{\partial \rho}$$

$$\frac{\partial f}{\partial l_2} = \frac{\partial f}{\partial \theta}\frac{d\theta}{dl_2} = \cancel{\theta}\,\frac{\partial f}{\partial \theta}\frac{1}{\rho}$$

$$\frac{\partial f}{\partial l_3} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial l_3} = \frac{\partial f}{\partial z}$$

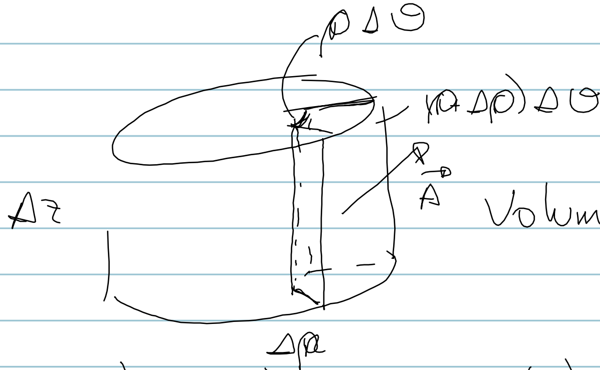
$$\vec{\nabla} f(\rho,\theta,z) = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

Em geral

$$\vec{\nabla} f(q_1,q_2,q_3) = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{q}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{q}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{q}_3$$

Divergente de $\vec{A}$

$$\lim_{V \to 0} \frac{\Phi}{Vol}$$



Volume $= \rho \Delta\rho \Delta\theta \Delta z$

$$\frac{\Phi_\rho}{\rho\Delta\rho\Delta\theta\Delta z} = \frac{V_\rho(\rho+\Delta\rho)(\rho+\Delta\rho)\cancel{\Delta z}\cancel{\Delta\theta} - V_\rho(\rho)\,\rho\,\cancel{\Delta z}\,\Delta\theta}{\rho\,\Delta\rho\,\cancel{\Delta z}\,\Delta\theta}$$

calculado em $\bar{\theta}, \bar{z}$

$$Vol \to 0 \longrightarrow \frac{\Phi_\rho}{Vol} = \cancel{\frac{1}{\rho}\left(\rho\frac{\partial V_\rho}{\partial\rho}\right)} \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho V_\rho\right)$$

$$\frac{\Phi_\theta}{Vol} = \frac{V_\theta(\theta+\Delta\theta)\cancel{\Delta\rho}\cancel{\Delta z} - V_\theta(\theta)\cancel{\Delta\rho}\cancel{\Delta z}}{\rho\,\cancel{\Delta\rho}\,\cancel{\Delta z}\,\Delta\theta}$$

calculado em $\bar{\rho}, \bar{z}$

$$Vol \to 0 \longrightarrow \frac{\Phi_\theta}{Vol} \to \frac{1}{\rho}\frac{\partial V_\theta}{\partial\theta}$$

$$\frac{\Phi_z}{Vol} = \frac{V_z(z+\Delta z)\cancel{\rho}\cancel{\Delta\theta}\cancel{\Delta\rho} - V_z(z)\cancel{\rho}\cancel{\Delta\theta}\cancel{\Delta\rho}}{\cancel{\rho}\,\cancel{\Delta\rho}\,\Delta z\,\cancel{\Delta\theta}}$$

calculado em $\bar{\rho}, \bar{\theta}$

$$Vol \to 0 \longrightarrow \frac{\Phi_z}{Vol} \to \frac{\partial V_z}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) \quad \cancel{\frac{1}{\rho}\left(\rho\frac{\partial V_\rho}{\partial\rho}\right)} + \frac{1}{\rho}\left(\frac{\partial V_\theta}{\partial\theta}\right) + \frac{\partial V_z}{\partial z}$$

O Laplaciano fica $\vec{\nabla}\cdot(\vec{\nabla}\varphi)$

$$\nabla^2\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Não vamos demonstrar ~~a~~ aqui, mas o rotacional pode ser obtido a partir de $\lim \oint \vec{V}\cdot d\vec{l}$

Área $\to 0$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_\rho & \rho V_\theta & V_z \end{vmatrix}$$

6.2

Coordenadas esféricas

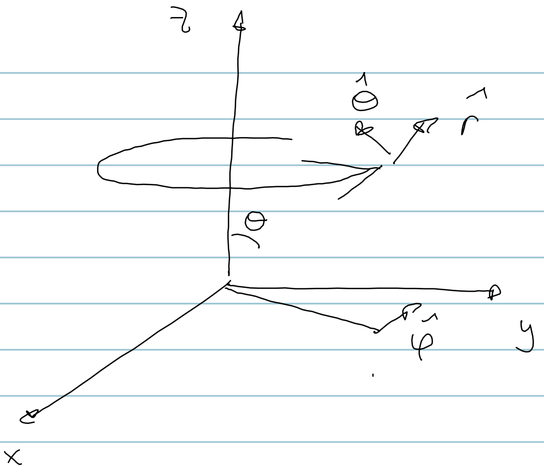
~~q_1~~ $q_1 \to r \to$ esferas centradas na origem

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$q_2 \to \theta$ cones fazendo ângulo θ com eixo z

$$\theta = \arccos\frac{z}{(x^2+y^2+z^2)^{1/2}}$$

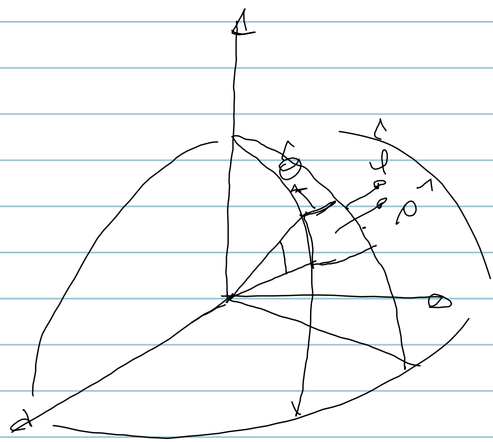
$q_3 \to \varphi$ meios-planos, ângulo φ com eixo x

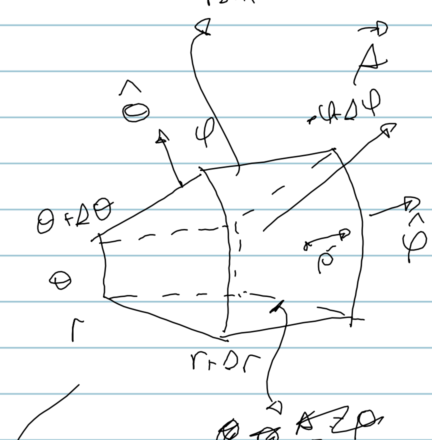


$dl_1 = dr$
$dl_2 = r\,d\theta$
$dl_3 = r\sin\theta\,d\varphi$
$h_1 = 1$; $h_2 = r$
$h_3 = r\sin\theta$
$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$

Gradiente $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat\varphi$

Divergente $\vec{A}$ $\lim\limits_{v\to 0} \frac{\phi}{v_0}$





$A = (r+\Delta r)^2\,\Delta\theta\,\Delta\varphi\,\sin$

$A = r\Delta\theta\Delta r$

$$\frac{\phi_r}{Vol} = \frac{A_r(r+\Delta r)(r+\Delta r)^2 \cancel{\sin\theta \Delta\theta \Delta\varphi} - A_r(r) r^2 \cancel{\sin\theta \Delta\theta \Delta\varphi}}{r^2 \cancel{\sin\theta} \Delta r \cancel{\Delta\theta \Delta\varphi}}$$

$$Vol \to 0 \quad \to \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)$$

$$\frac{\phi_\theta}{Vol} = \frac{A_\theta(\theta+\Delta\theta) r \sin(\theta+\Delta\theta) \Delta r \Delta\varphi - A_\theta(\theta) r \sin\theta \Delta r \Delta\varphi}{r^2 \sin\theta \Delta r \Delta\theta \Delta\varphi}$$

$$Vol \to 0 \quad \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)$$

$$\frac{\phi_\varphi}{Vol} = \frac{A_\varphi(\varphi+\Delta\varphi) r \Delta\theta \Delta r - A_\varphi(\varphi) r \Delta\theta \Delta r}{r^2 \sin\theta \Delta r \Delta\theta \Delta\varphi}$$

$$Vol \to 0 \quad \frac{1}{r \sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2 A_r\right) + \cancel{r\frac{\partial A_\theta}{\partial\theta}} + r\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + r\frac{\partial A_\varphi}{\partial\varphi}\right]$$

Obtemos o Laplaciano, substituindo o gradiente

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} f = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right]$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ V_r & rV_\theta & r\sin\theta V_\varphi \end{vmatrix}$$

Exercícios

① Seja $\vec{A}$ um vetor constante e $\vec{r}$ o vetor que vai da origem ao ponto (x, y, z)
Mostre que $(\vec{r} - \vec{A}) \cdot \vec{r} = 0$ é a equação esferas centradas em $\vec{A}/2$

② Mostre que $\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\vec{\nabla}\varphi \cdot \vec{\nabla}\psi$

③ Mostre que $\vec{\nabla}(\vec{A} \cdot \vec{r}) = \vec{A}$
$(\vec{u} \cdot \vec{\nabla})\vec{r} = \vec{u}$

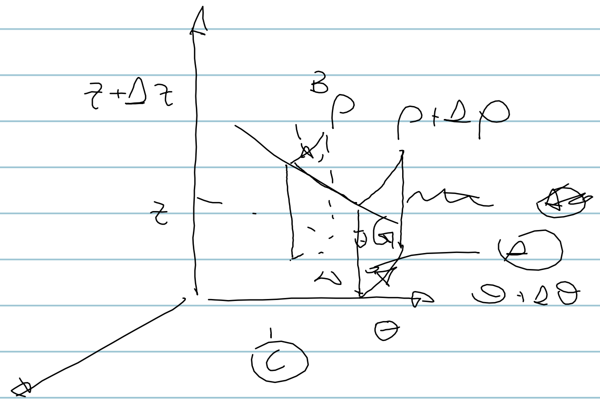
④ Mostre que $\vec{\nabla} \times (f\vec{A}) = \vec{\nabla} f \times \vec{A} + f(\vec{\nabla} \times \vec{A})$

⑤ Uma das eq. de Maxwell é $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\vec{E}$ (V/m) e $\vec{B}$ (Tesla)
Mostre que $\left| \oint \vec{E} \cdot d\vec{\ell} \right|$ é $20\,\mu V$

num quadrado de 10 cm de lado se $\vec{B}$ é $2 \times 10^{-3} t$ Tesla

⑥ Ache a velocidade $\vec{v}$ em coordenadas ~~polar~~ esféricas.

⑦ Obter a expressão do rotacional em coordenadas cilíndricas. (Tentem repetir)



$$\vec{dA} = \rho\,\Delta\theta\,\Delta r\,\hat{z} + \rho\,\Delta\theta\,\Delta z\,\hat{\rho} + \Delta r\,\Delta z\,\hat{\theta}$$

A

$\perp \hat{z}$ $\quad \frac{\oint}{\text{Area}} = \frac{\oint}{\rho\Delta\theta\Delta z} = \frac{V_\theta(\rho+\Delta\rho)(\rho+\Delta\rho)\Delta\theta - V_\theta(\rho)\rho\Delta\theta}{\rho\Delta\theta\Delta z} \frac{-V_\rho(\theta+\Delta\theta)\Delta\rho + V_\rho(\theta)\Delta\rho}{\rho\Delta\theta\Delta z}$

$$\lim \to \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho V_\theta) - \frac{\partial V_\rho}{\partial\theta}\right]$$

B

$\perp \hat{\theta}$ $\quad \frac{\oint}{\text{Área}} = \frac{\oint}{\Delta r\,\Delta z} = \frac{V_\rho(z+\Delta z)\Delta\rho - V_\rho(z)\Delta\rho + V_z(\rho)\Delta z - V_z(\rho+\Delta\rho)\Delta z}{\Delta r\,\Delta z}$

$$\lim \to \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial\rho}\right)$$

C

$\perp \hat{\rho}$ $\quad \frac{\oint}{\rho\Delta\theta\Delta z} = \frac{V_z(\theta+\Delta\theta)\Delta z - V_z(\theta)\Delta z + V_\theta(z)\rho\Delta\theta - V_\theta(z+\Delta z)\rho\Delta\theta}{\rho\Delta\theta\Delta z}$

$$\lim \to \frac{1}{\rho}\left(\frac{\partial V_z}{\partial \theta} - \rho \frac{\partial V_\theta}{\partial z}\right) =$$

$$= \frac{1}{\rho}\left(\frac{\partial V_z}{\partial \theta} - \frac{\partial}{\partial z}(\rho V_\theta)\right)$$

$$\vec{\nabla} \times \vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\theta & A_z \end{vmatrix}$$

Finalmente, algumas observações sobre o operadores e funções de $\vec{r} - \vec{r}' \equiv \vec{\mathscr{r}}$

Se fizermos uma mudança de referencial

$x \to x - x'$ $\qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x - x'}$

$y \to y - y'$

$z \to z - z'$

Então as relações que mostramos para funções que dependem somente de $|\vec{r}|$ e como produto com $\vec{r}$, valem para $\vec{r} - \vec{r}'$

Das páginas 26 e 27

$$\vec{\nabla} g(|\vec{r} - \vec{r}'|) = \vec{\nabla} g(|\vec{\mathscr{r}}|) = \frac{\vec{\mathscr{r}}}{\mathscr{r}} \frac{dg}{d\mathscr{r}}$$

$$\vec{\nabla} \cdot [g(\mathscr{r})\vec{\mathscr{r}}] = \mathscr{r}\frac{dg(\mathscr{r})}{d\mathscr{r}} + 3g(\mathscr{r})$$

$\vec{\nabla} \times (\vec{r}) = 0$
$\vec{\nabla} \cdot \vec{r} = 3$

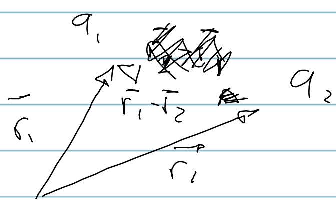
<u>Eletrostática</u>

carga → propriedade da matéria
→ carga total conserva
→ Dois tipos de carga $\oplus$ e $\ominus$
→ unidade coulomb — C
→ carga de 1 elétron $1{,}6 \times 10^{-19}$ C

~~cargas em~~ $\vec{r}_1$ Dada uma carga q_1 em $\vec{r}_1$ e q_2 em $\vec{r}_2$, surge uma força dada por:

$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$, força sobre q_1

$$\vec{F}_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1)$$, força sobre q_2



$\epsilon_0 = 8{,}85 \times 10^{-12} \frac{C^2}{N m^2}$

permissividade do espaço livre (vácuo)

Se temos uma distribuição de cargas em $\vec{r}_i'$ com carga q_i' e queremos obter a força