Random number generators

"Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin."

John von Neumann, 1951.

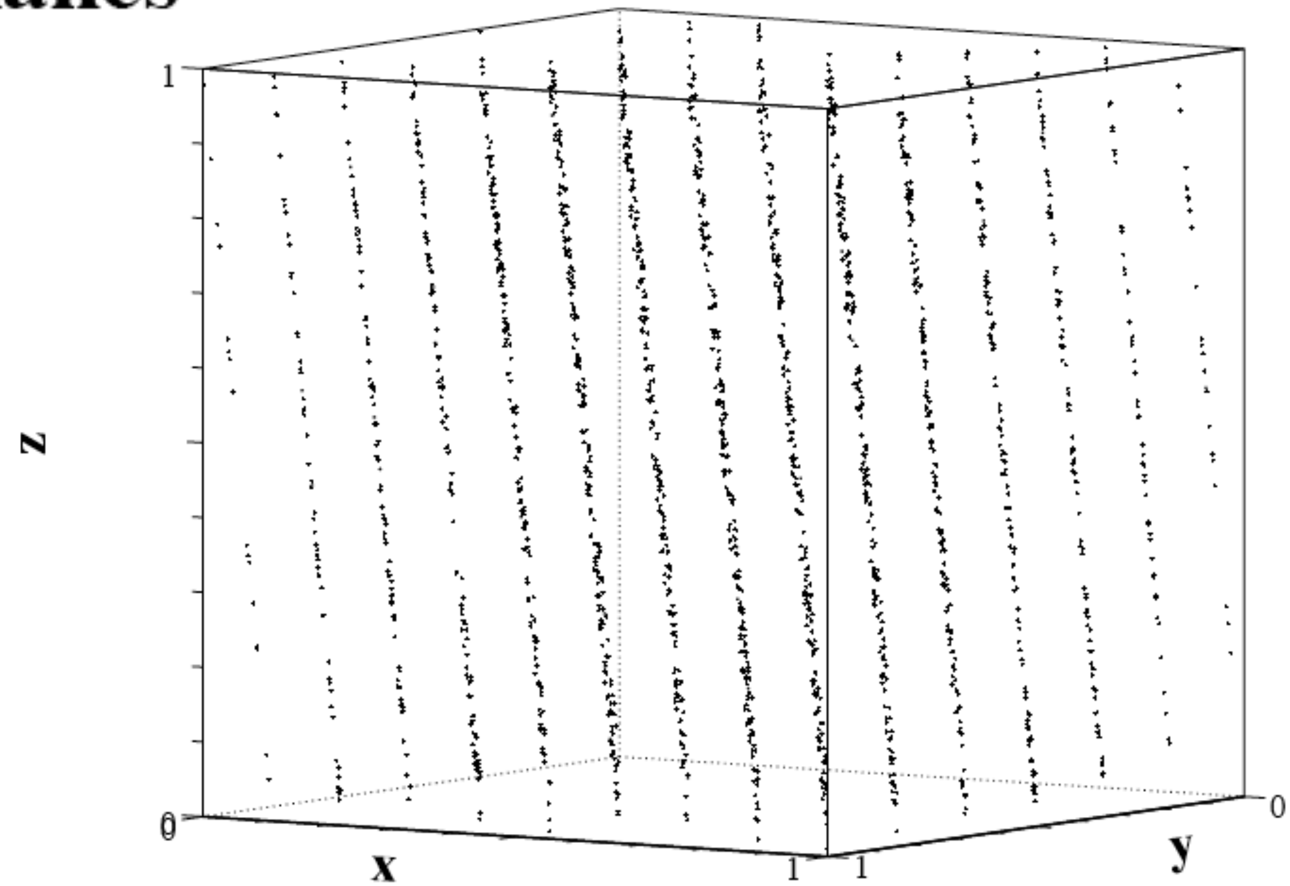
Thus, we will talk about PSEUDO-random number generators.

Random number generators

- They are the soul of a Monte Carlo code
- They imitate the estocastic nature of particle interactions.
- Do not ever use language-intrinsic RNG in “serious” MC simulations
- You should use well-known and tested RNG

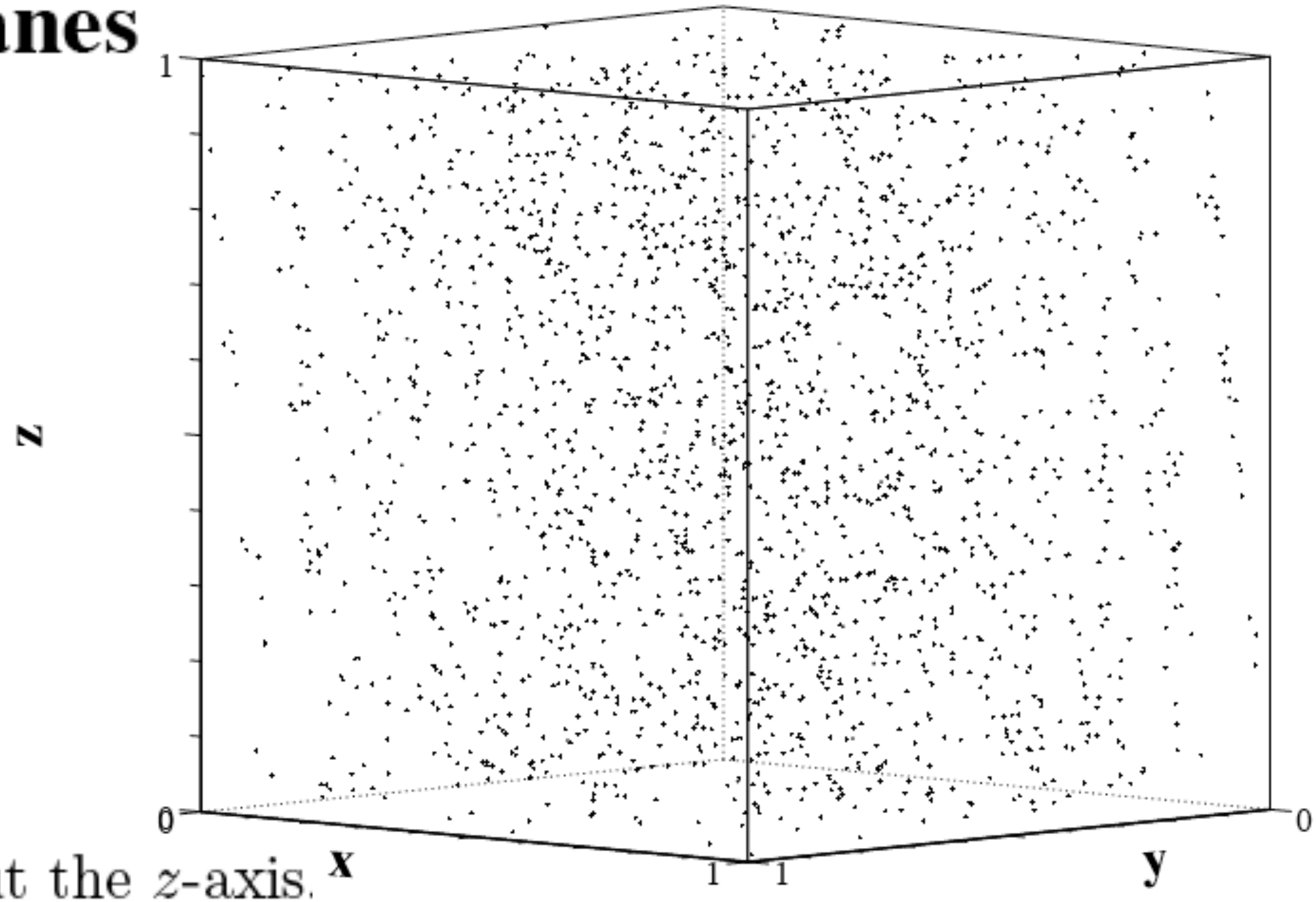
Random number generators

Marsaglia planes



Random number generators

Marsaglia planes



rotated by 10° about the z-axis.

Random number generators

Linear congruential RNG (LCRNG)

$$X_{n+1} = \mathrm{mod}(aX_n + c, 2^{32})$$

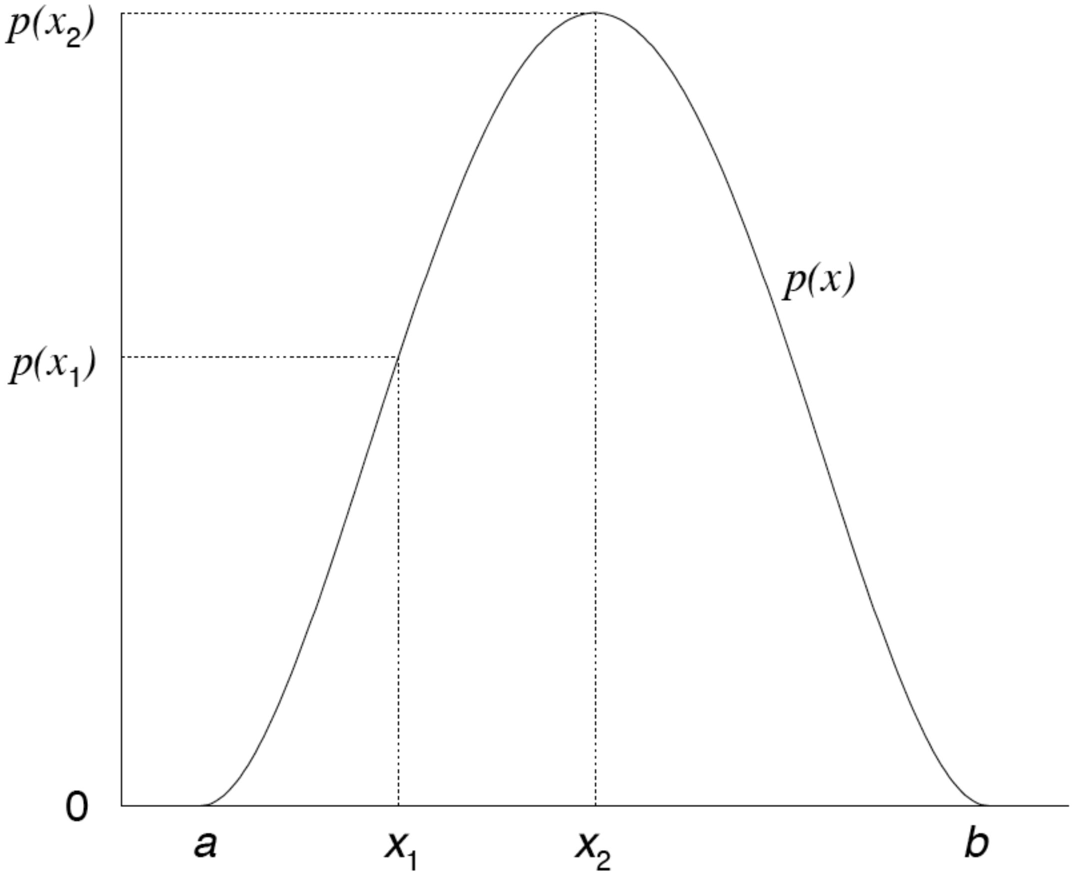
$$r_n = 1/2 + X_n/2^{32}$$

Period of 2^{32} for odd c.

Period of 2^{64} for 64-bit integer numbers.

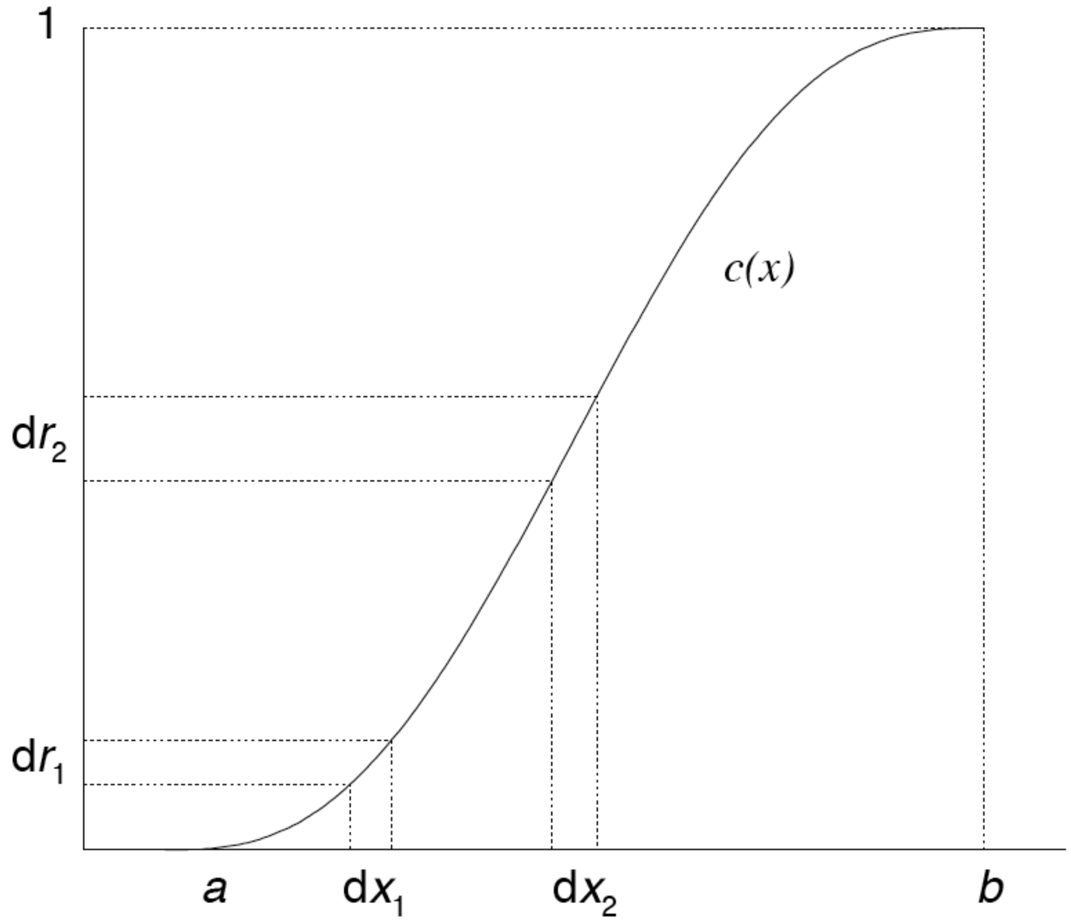
Sampling Methods

Probability density function



Sampling Methods

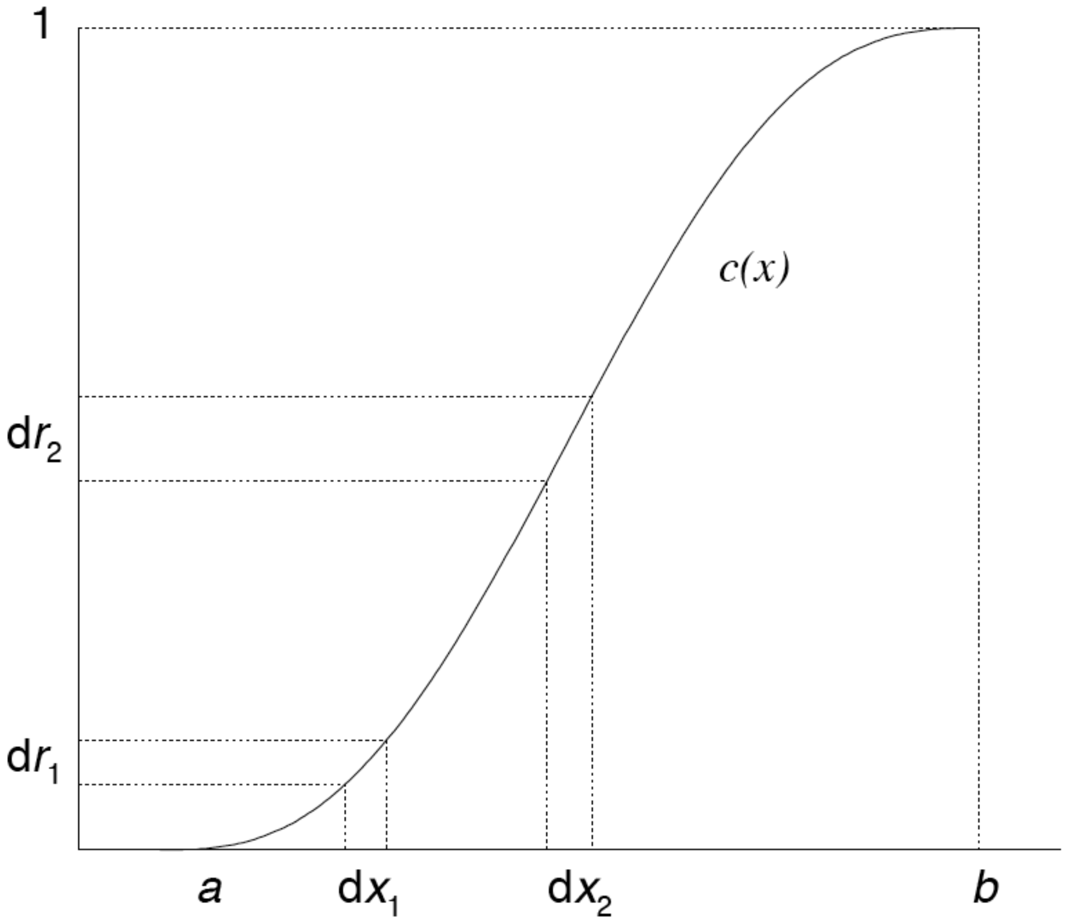
Cumulative probability function.



Sampling Methods

Cumulative probability function

$$\frac{\mathrm{d}r_1}{\mathrm{d}r_2} = \frac{(\mathrm{d}/\mathrm{d}x)c(x)|_{x=x_1}}{(\mathrm{d}/\mathrm{d}x)c(x)|_{x=x_2}} = \frac{p(x_1)}{p(x_2)}$$



Sampling Methods

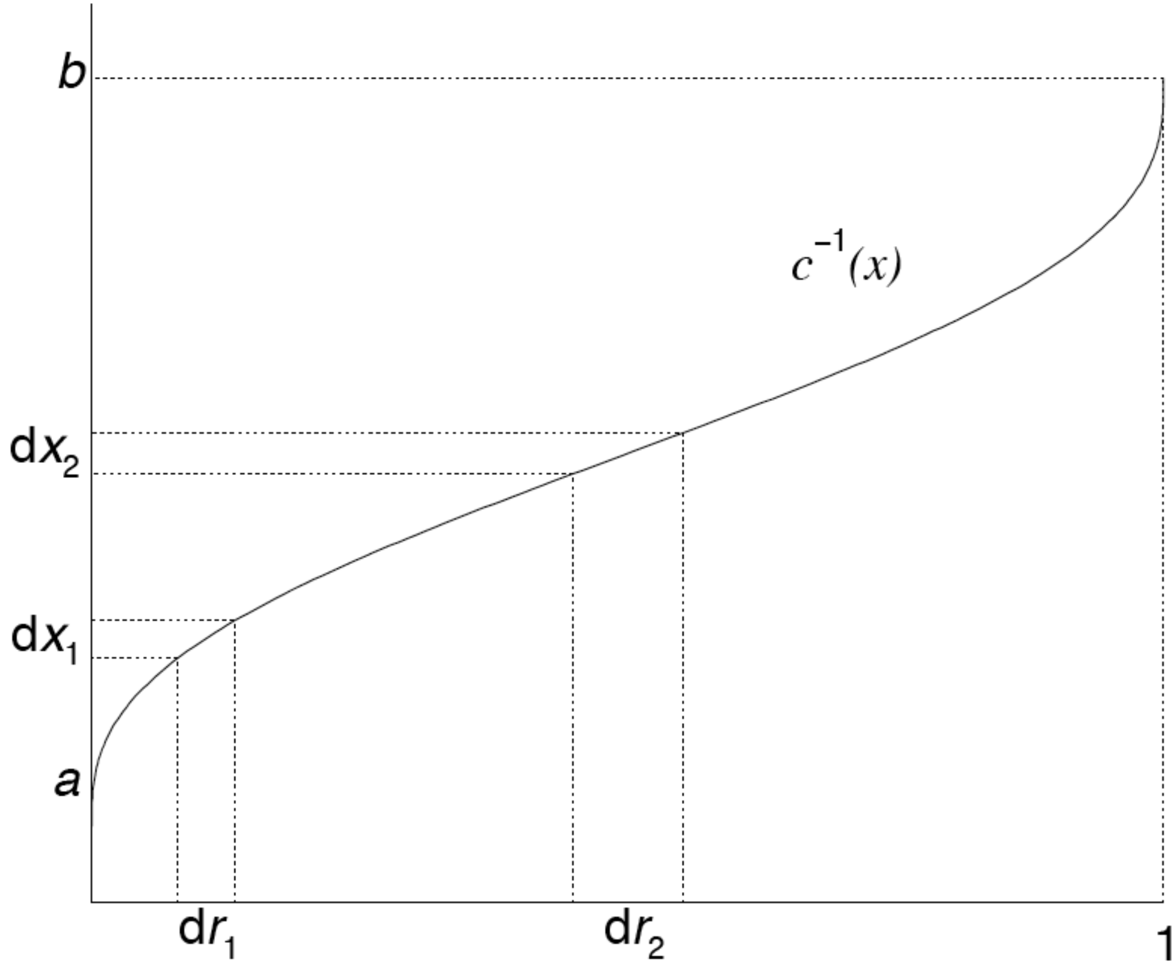
Direct method.
The distribution is inverted

$$x = c^{-1}(r)$$

A random number is generated and it is used to obtain the corresponding stochastic variable value.

Sampling Methods

Inverted cumulative probability function



Sampling Methods

Rejection Method.

The distribution is scaled according to its maximum value

$$f(x) = p(x)/p(x_{\max})$$

1- A random number r_1 is generated to get x

$$x = a + (b - a)r_1$$

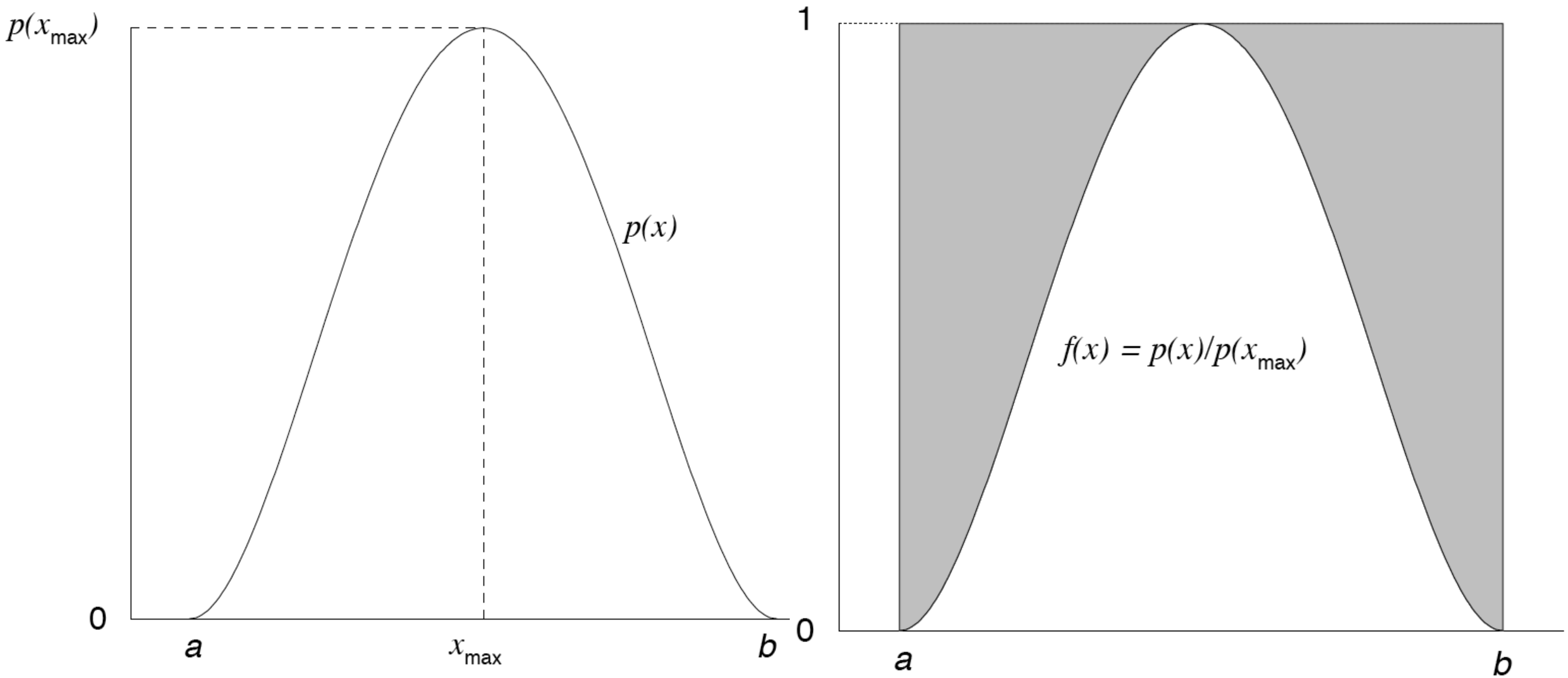
2- Another random number r_2 is generated

3- x is accepted if:

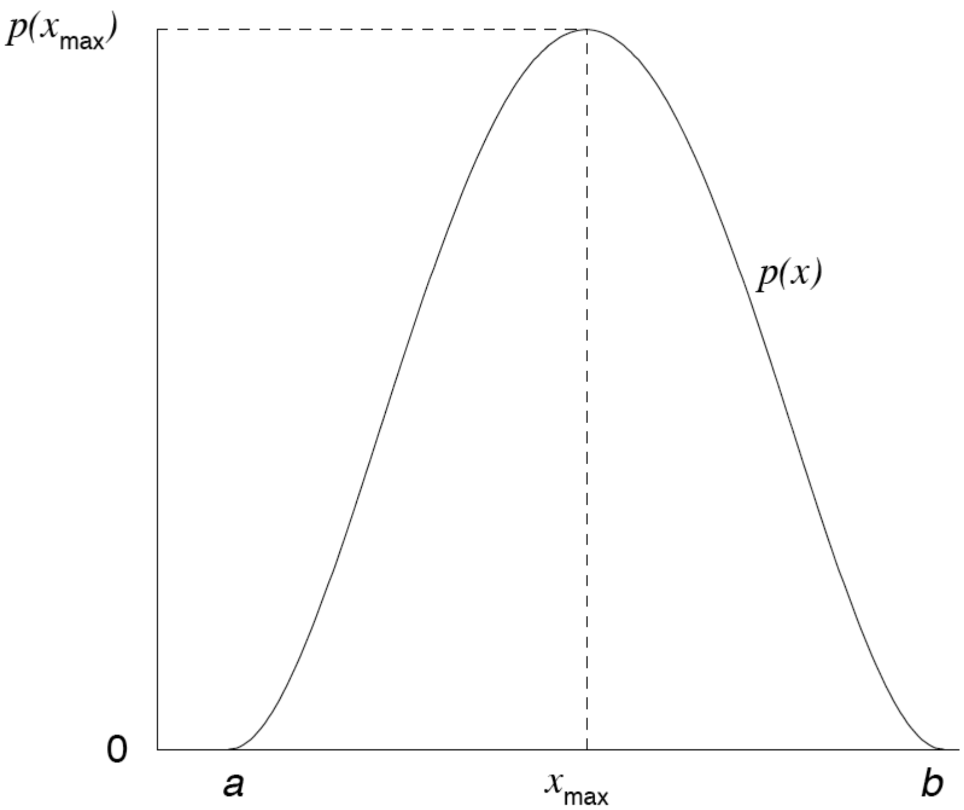
$$r_2 < p(x)/p(x_{\max})$$

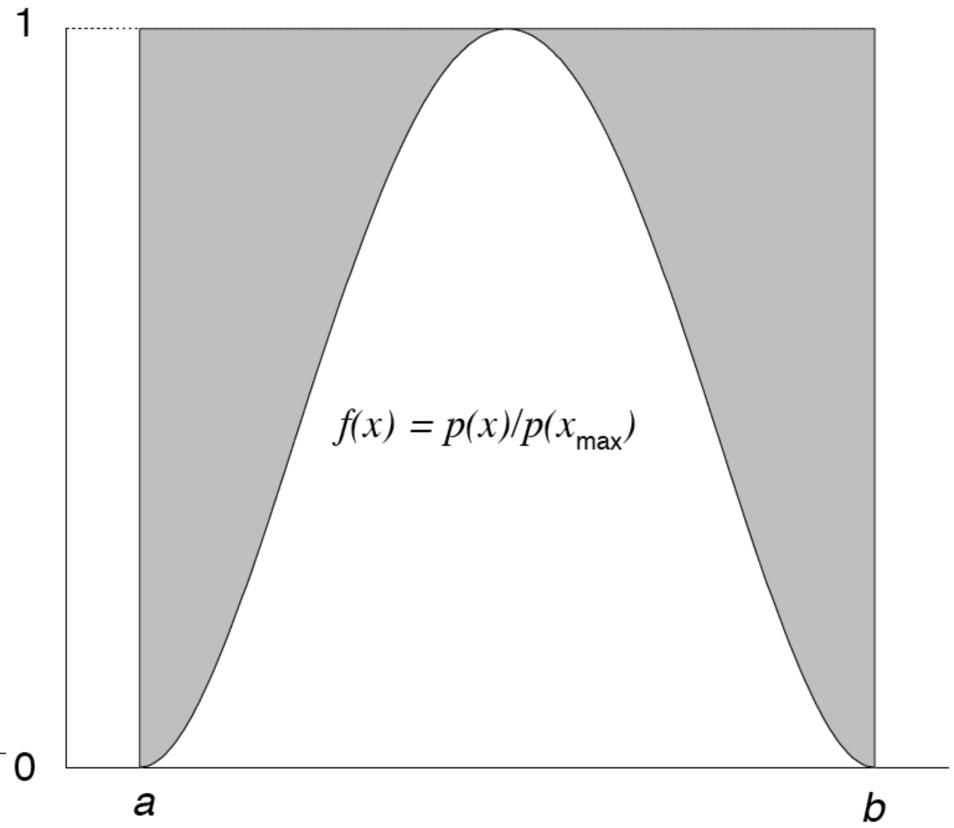
else, back to step 1.

Sampling Methods



Sampling Methods





Sampling Methods

Mixed method.

Probability density function

$$p(x)\mathrm{d}x = Ne^{-x^2}\frac{2x\mathrm{d}x}{(1+x^2)^2} \qquad 0 \le x < \infty$$

Devide it into two parts.

First one:

$$f(x)\mathrm{d}x = \frac{2x\mathrm{d}x}{(1+x^2)^2}\mathrm{d}x \qquad 0 \le x < \infty$$

$$r = c(x) = 1 - \frac{1}{1+x^2} \quad \Rightarrow \quad x = \sqrt{\frac{r}{1-r}}$$

Sampling Methods

Mixed method (cont.)

This is equivalent to:

$$u = 1 - \frac{1}{1+x^2} \qquad x = \sqrt{\frac{u}{1-u}}$$

Second part:

$$g(x)\mathrm{d}x = \exp\left(-\frac{u}{1-u}\right)\mathrm{d}u \qquad 0 \le u \le 1$$

Then apply rejection method to this part:

Sampling Methods

Examples.

Cylindrical beam.

$$p(\rho, \phi)\ \mathrm{d}\rho\ \mathrm{d}\phi = \frac{1}{\pi \rho_0^2} \rho\ \mathrm{d}\rho\ \mathrm{d}\phi \qquad 0 \le \rho \le \rho_0 \quad 0 \le \phi \le 2\pi$$

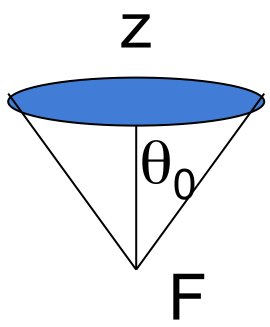
$$p(\rho, \phi)\ \mathrm{d}\rho\ \mathrm{d}\phi = \mathrm{d}p_1(\rho)\ \mathrm{d}p_2(\phi)$$

$$p_2(\phi)\ \mathrm{d}\phi = \frac{1}{2\pi}\ \mathrm{d}\phi \qquad 0 \le \phi \le 2\pi$$

After the inversion: $\rho = \rho_0 \sqrt{r_1}$ $\qquad \phi = 2\pi r_2$

Sampling Methods

Examples.



Collimated isotropic point source.

$$p(\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi = \frac{\mathrm{d}\phi}{2\pi}\frac{\sin\theta\ \mathrm{d}\theta}{1-\cos\theta_0} \qquad 0 \le \theta \le \theta_0 \quad 0 \le \phi \le 2\pi$$

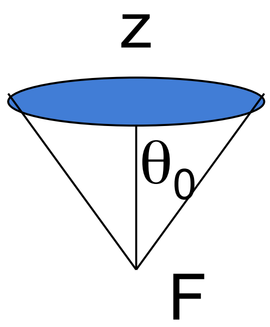
$$p(\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi = p_1(\theta)\mathrm{d}\theta\ p_2(\phi)\mathrm{d}\phi$$

$$p_1(\theta)\mathrm{d}\theta = \frac{\sin\theta\mathrm{d}\theta}{1-\cos\theta_0} \qquad 0 \le \theta \le \theta_0$$

$$p_2(\phi)\mathrm{d}\phi = \frac{1}{2\pi}\mathrm{d}\phi \qquad 0 \le \phi \le 2\pi$$

Sampling Methods

Example.



Collimated isotropic point source (cont).

$$c_1(\theta) = \frac{1}{1-\cos\theta_0}\int_0^{\theta} \sin\theta' \mathrm{d}\theta' = \frac{1-\cos\theta}{1-\cos\theta_0}$$

$$c_2(\phi) = c_2(\phi) = \frac{1}{2\pi}\int_0^{\phi} \mathrm{d}\phi' = \frac{\phi}{2\pi}$$

Then, inversion:

$$\cos\theta = 1 - r_1[1-\cos\theta_0]$$

$$\phi = 2\pi r_2$$

Sampling Methods

Example. Two-variable distribution

Distribution. $$p(x,y)\ \mathrm{d}x\ \mathrm{d}y = (x+y)\ \mathrm{d}x\ \mathrm{d}y \qquad 0 \leq x, y \leq 1$$

Marginal probability $$m(x) = \int_0^1 \mathrm{d}y\ (x+y) = x + \frac{1}{2}$$

Conditional probability $$p(y|x) = \frac{p(x,y)}{m(x)} = \frac{x+y}{x+\frac{1}{2}}$$

Determine x.

$$r_1 = c(x) = \int_0^x \mathrm{d}x'\ \left(x' + \frac{1}{2}\right) = \frac{x^2}{2} + \frac{x}{2} \qquad x = \frac{-1+\sqrt{1+8r_1}}{2}$$

Sampling Methods

Example. Two-variable distribution (cont.)

After having determined x, let's find y using the conditional probability

$$r_2 = c(y|x) = \int_0^y \mathrm{d}y \; p(y|x) = \frac{y^2 + 2xy}{2x+1}$$

$$y = -x + \sqrt{x^2 + r_2(2x+1)}$$