Geometrical analysis

Description of the particle geometrical phase-space:
position and direction.

$$\{\vec{x}, \vec{u}\}$$

$$\vec{x} = (x, y, z)$$

$$\vec{u} = (u, v, w) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Geometrical analysis

Particle displacement:

$$\vec{x} = \vec{x}_0 + \vec{u}_0 s.$$

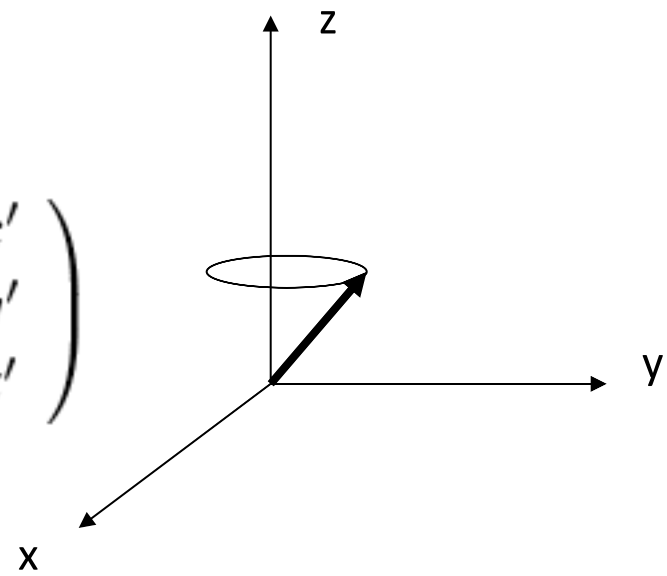
$$\begin{aligned} x &= x_0 + u_0 s \\ y &= y_0 + v_0 s \\ z &= z_0 + w_0 s \end{aligned}$$

$$\mathcal{T}(\{\vec{x}_0, \vec{u}_0\}, s) = \{\vec{x}_0 + \vec{u}_0 s, \vec{u}_0\} = \{\vec{x}, \vec{u}_0\}$$

Geometrical analysis.

Rotation by angle ϕ around z axis.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$



Geometrical analysis.

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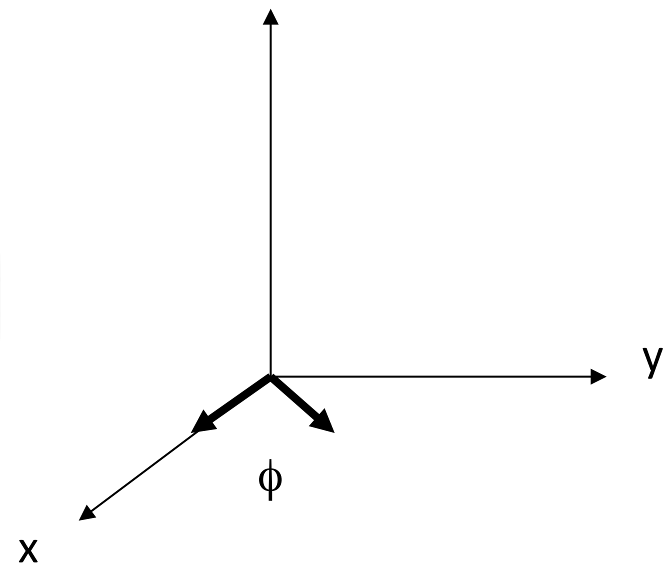
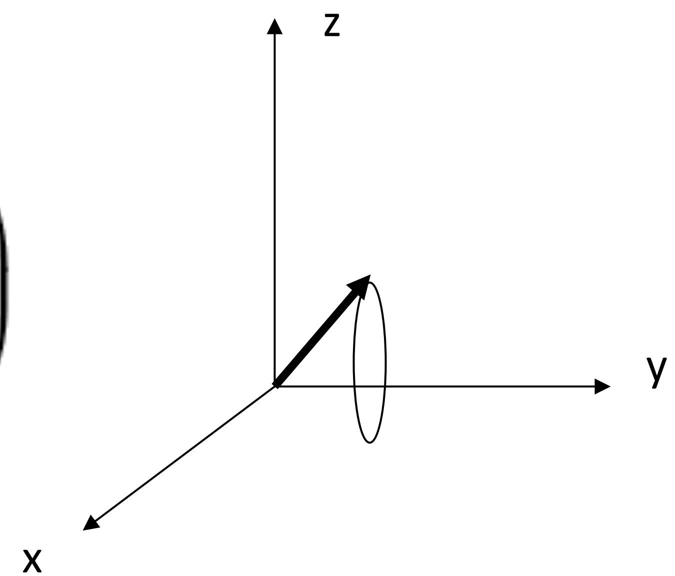


Geometrical analysis.

Rotation by angle θ around y axis.

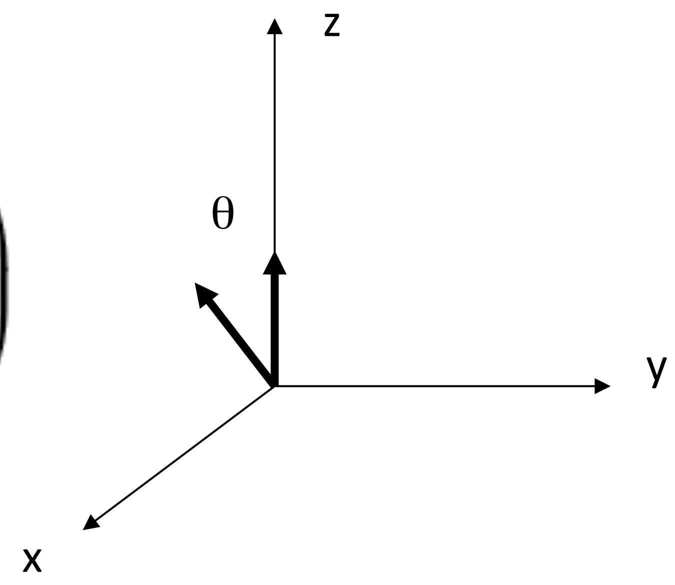
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$



Geometrical analysis.

Rotation by angle θ around y axis.

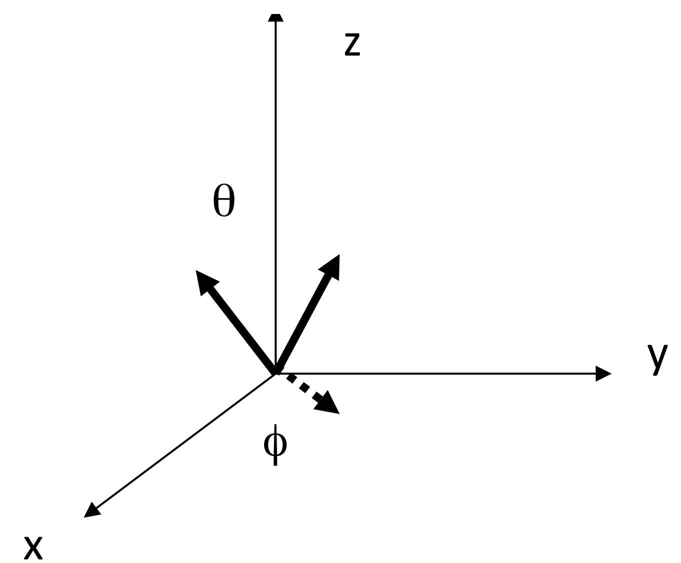
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$



Geometrical analysis.

Coordinate transformation between rotated systems

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$



Geometrical analysis.

Rotated-to-laboratory transformation operator

$$\Re(\theta,\phi) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\|\Re(\theta,\phi)\| = 1 \qquad \Re^{-1}(\theta,\phi) = \Re^{\mathrm{T}}(\theta,\phi)$$

Geometrical analysis.

Change of particle direction

$$\vec{u}_0 = u_0, v_0, w_0) = (\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)$$

$$\vec{u} = \Re(\theta_0, \phi_0)\Re(\Theta, \Phi)\Re^{-1}(\theta_0, \phi_0)\vec{u}_0$$

Lab-to-particle system rotation

$$\Re^{-1}(\theta_0, \phi_0)\vec{u}_0 = \hat{z} = (0, 0, 1)$$

Direction change in the particle system

$$\Re(\Theta, \Phi)\hat{z} = (\sin\Theta \cos\Phi, \sin\Theta \sin\Phi, \cos\Theta)$$

Geometrical analysis.

Change of particle direction (cont.)

Finally, coordinates of particles are transformed from the particle to the lab system

$$\Re(\theta, \phi)\Re(\Theta, \Phi)\hat{z}$$

Geometrical analysis.

Change of particle direction (cont.)

Explicitly

$$
\begin{aligned}
u = \sin\theta\cos\phi &= u_0\cos\Theta + \sin\Theta(w_0\cos\Phi\cos\phi_0 - \sin\Phi\sin\phi_0) \\
v = \sin\theta\sin\phi &= v_0\cos\Theta + \sin\Theta(w_0\cos\Phi\sin\phi_0 + \sin\Phi\cos\phi_0) \\
w = \cos\theta &= w_0\cos\Theta - \sin\Theta\sin\theta_0\cos\Phi
\end{aligned}
$$

$$\mathcal{R}(\{\vec{x}_0, \vec{u_0}\}, \Theta, \Phi) = \{\vec{x}_0, \Re(\theta_0, \phi_0)\Re(\Theta, \Phi)\Re^{-1}(\theta_0, \phi_0)\vec{u}_0\} = \{\vec{x}_0, \vec{u}\}$$

Geometrical analysis.

The trajectory of the particle is constructed by combining translations
and rotations

$$(s_1, s_2, s_3 \cdots)$$

Displacements between interactions

Deflections

$$([\Theta_1, \Phi_1], [\Theta_2, \Phi_2], [\Theta_3, \Phi_3] \cdots)$$

Geometrical analysis.

Unimos traslaciones con rotaciones para conformar la trayectoria de la partícula (cont.)

$$
\begin{aligned}
\{\vec{x}_1, \vec{u}_0\} &= \mathcal{T}(\{\vec{x}_0, \vec{u}_0\}, s_1) \\
\{\vec{x}_1, \vec{u}_1\} &= \mathcal{R}(\{\vec{x}_1, \vec{u}_0\}, \Theta_1, \Phi_1) \\
\{\vec{x}_2, \vec{u}_1\} &= \mathcal{T}(\{\vec{x}_1, \vec{u}_1\}, s_2) \\
\{\vec{x}_2, \vec{u}_2\} &= \mathcal{R}(\{\vec{x}_2, \vec{u}_1\}, \Theta_2, \Phi_2) \\
\{\vec{x}_3, \vec{u}_2\} &= \mathcal{T}(\{\vec{x}_2, \vec{u}_2\}, s_3) \\
\{\vec{x}_3, \vec{u}_3\} &= \mathcal{R}(\{\vec{x}_3, \vec{u}_2\}, \Theta_3, \Phi_3) \\
&\vdots
\end{aligned}
$$