Implementação do método FDTD para solução das Equações de Maxwell com UPML com condição de contorno

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8 de dezembro de 2011

Resumo

Este documento serve como referencia das equações e dos procedimentos usados na implementação de Uniaxial Perfect Matched Layer UPML nos algoritmos FDTD para paralelismo em GPU.

1 Introdução

A técnica UPML consiste em considerar um material anisotrópico como contorno do domínio computacional e escolher as componentes adequadas dos tensores permissividade elétrica e permeabilidade magnética. Desta forma obtemos contornos que não refletem ondas planas para qualquer ângulo incidente e absorvem estas ondas fazendo com que os campos que decaiam exponencialmente com a distância. Os detalhes desta técnica podem ser encontrados no Taflove 2a edição no capítulo 7. Inicialmente vamos considerar as equações de Faraday (2) e Ampere (1) onde foi aplicada a transformadas de Fourier temporal nos campos:

$$\nabla \times \hat{\mathbf{H}} = j\omega\epsilon \, \bar{\mathbf{S}} \, \hat{\mathbf{E}} \,, \tag{1}$$

$$\nabla \times \hat{\mathbf{E}} = j\omega\mu \, \bar{\mathbf{S}} \, \hat{\mathbf{H}} \,, \tag{2}$$

onde $\hat{\mathbf{E}}$ e $\hat{\mathbf{H}}$ são as transformadas de Fourier do campo elétrico e magnético respectivamente, ϵ é a permissividade elétrica e μ a permeabilidade magnética do meio, $\bar{\mathbf{S}}$ é o tensor que proporciona a anisotropia das propriedades eletromagnéticas.

O tensor $\bar{\mathbf{S}}$ escolhido para minimizar a reflexão para ondas incidentes é dado pela equação (3),

$$\bar{\mathbf{S}} = \begin{Vmatrix} \frac{s_y s_z}{s_x} & & \\ & \frac{s_x s_z}{s_y} & \\ & & \frac{s_x s_y}{s_z} \end{Vmatrix} \,, \tag{3}$$

onde,

$$s_l = \kappa_l + \frac{\sigma_l}{j\omega\epsilon} \,, \tag{4}$$

com $l = x, y, z$. O valores dos parâmetros κ_l e σ_l serão descritos mais adiante, ω é o termo de frequência da transformada de Fourier.

As definições das relações constitutivas, dadas pelas equações (5-10), são escolhidas de forma a evitar constantes dependentes de ω e assim evitar a necessidade de convoluções.

$$\hat{\mathbf{D}}_{\mathbf{x}} = \frac{\epsilon s_z}{s_x}\hat{\mathbf{E}}_{\mathbf{x}}\,, \tag{5}$$

$$\hat{\mathbf{D}}_{\mathbf{y}} = \frac{\epsilon s_x}{s_y}\hat{\mathbf{E}}_{\mathbf{y}}\,, \tag{6}$$

$$\hat{\mathbf{D}}_{\mathbf{z}} = \frac{\epsilon s_y}{z_x}\hat{\mathbf{E}}_{\mathbf{z}}\,, \tag{7}$$

$$\hat{\mathbf{B}}_{\mathbf{x}} = \frac{\mu s_z}{s_x}\hat{\mathbf{H}}_{\mathbf{x}}\,, \tag{8}$$

$$\hat{\mathbf{B}}_{\mathbf{y}} = \frac{\mu s_x}{s_y}\hat{\mathbf{H}}_{\mathbf{y}}\,, \tag{9}$$

$$\hat{\mathbf{B}}_{\mathbf{z}} = \frac{\mu s_y}{s_z}\hat{\mathbf{H}}_{\mathbf{z}}\,. \tag{10}$$

Substituindo as equações (5-10) em (2) e (1) obtemos,

$$\begin{vmatrix} \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z} \\ \frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x} \\ \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} \end{vmatrix} = i\omega \begin{Vmatrix} s_y & & \\ & s_z & \\ & & s_x \end{Vmatrix} \begin{vmatrix} \hat{D}_x \\ \hat{D}_y \\ \hat{D}_z \end{vmatrix} \tag{11}$$

$$\begin{vmatrix} \frac{\partial \hat{E}_z}{\partial y} - \frac{\partial \hat{E}_y}{\partial z} \\ \frac{\partial \hat{E}_x}{\partial z} - \frac{\partial \hat{E}_z}{\partial x} \\ \frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} \end{vmatrix} = -j\omega \begin{Vmatrix} s_y & & \\ & s_z & \\ & & s_x \end{Vmatrix} \begin{vmatrix} \hat{H}_x \\ \hat{H}_y \\ \hat{H}_z \end{vmatrix} \tag{12}$$

Usando a equação (4) em e (11), (12) e tomando a transformada inversa de Fourier,

$$\begin{vmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{vmatrix} = \frac{\partial}{\partial t} \begin{Vmatrix} \kappa_y & & \\ & \kappa_z & \\ & & \kappa_x \end{Vmatrix} \begin{vmatrix} D_x \\ D_y \\ D_z \end{vmatrix} + \frac{1}{\epsilon} \begin{Vmatrix} \sigma_y & & \\ & \sigma_z & \\ & & \sigma_x \end{Vmatrix} \begin{vmatrix} D_x \\ D_y \\ D_z \end{vmatrix} \tag{13}$$

$$\left|\begin{matrix} \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \\ \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} \end{matrix}\right| = -\frac{\partial}{\partial t}\left\|\begin{matrix} \kappa_y & & \\ & \kappa_z & \\ & & \kappa_x \end{matrix}\right\| \left|\begin{matrix} B_x \\ B_y \\ B_z \end{matrix}\right| - \frac{1}{\epsilon}\left\|\begin{matrix} \sigma_y & & \\ & \sigma_z & \\ & & \sigma_x \end{matrix}\right\| \left|\begin{matrix} B_x \\ B_y \\ B_z \end{matrix}\right| \tag{14}$$

2 Caso 2-d TEz

As derivadas em se anulam. As componentes das equações (13) e (14) necessárias são:

$$\frac{\partial H_z}{\partial y} = \kappa_y \frac{\partial D_x}{\partial t} + \frac{\sigma_y}{\epsilon} D_x \tag{15}$$

$$\frac{\partial H_z}{\partial x} = -\kappa_z \frac{\partial D_y}{\partial t} - \frac{\sigma_z}{\epsilon} D_y \tag{16}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\kappa_x \frac{\partial B_z}{\partial t} - \frac{\sigma_x}{\epsilon} B_z \tag{17}$$

As equações são discretizadas segundo o esquema de Yee:

$$\begin{aligned} \frac{H_z|^n_{i,j+1} - H_z|^n_{i,j}}{\Delta y} &= \kappa_y \frac{D_x|^{n+1/2}_{i,j+1/2} - D_x|^{n-1/2}_{i,j+1/2}}{\Delta t} \\ &+ \frac{\sigma_y}{\epsilon} \frac{D_x|^{n+1/2}_{i,j+1/2} + D_x|^{n-1/2}_{i,j+1/2}}{2} , \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{H_z|^n_{i+1,j} - H_z|^n_{i,j}}{\Delta x} &= -\kappa_z \frac{D_y|^{n+1/2}_{i+1/2,j} - D_y|^{n-1/2}_{i+1/2,j}}{\Delta t} \\ &- \frac{\sigma_z}{\epsilon} \frac{D_y|^{n+1/2}_{i+1/2,j} + D_y|^{n-1/2}_{i+1/2,j}}{2} , \end{aligned} \tag{19}$$

$$\begin{aligned} &\frac{E_y|^{n+1/2}_{i+1/2,j} - E_y|^{n+1/2}_{i-1/2,j}}{\Delta x} - \frac{E_x|^{n+1/2}_{i,j+1/2} - E_x|^{n+1/2}_{i,j-1/2}}{\Delta y} \\ &= \kappa_x \frac{B_z|^{n+1}_{i,j} - B_z|^n_{i,j}}{\Delta t} + \frac{\sigma_x}{\epsilon} \frac{B_z|^{n+1}_{i,j} + B_z|^n_{i,j}}{2} . \end{aligned} \tag{20}$$

Devemos discretizar também as relações constitutivas (5), (6) e (10), substituindo (4) nessas relações e tomando a transformada de Fourier inversa, obtemos:

$$\kappa_x \frac{\partial D_x}{\partial t} + \frac{\sigma_x}{\epsilon} D_x = \epsilon \left(\kappa_z \frac{\partial E_x}{\partial t} + \frac{\sigma_z}{\epsilon} E_x \right) , \tag{21}$$

$$\kappa_y \frac{\partial D_y}{\partial t} + \frac{\sigma_y}{\epsilon} D_y = \epsilon \left(\kappa_x \frac{\partial E_y}{\partial t} + \frac{\sigma_x}{\epsilon} E_y \right) , \tag{22}$$

$$\kappa_z \frac{\partial B_z}{\partial t} + \frac{\sigma_z}{\epsilon} H_z = \mu \left(\kappa_y \frac{\partial H_z}{\partial t} + \frac{\sigma_y}{\epsilon} H_z \right) . \tag{23}$$

Discretizando usando novamente o esquema de Yee:

$$\begin{aligned} &\kappa_x \frac{D_x|_{i,j+1/2}^{n+1/2} - D_x|_{i,j+1/2}^{n-1/2}}{\Delta t} + \frac{\sigma_x}{\epsilon} \frac{D_x|_{i,j+1/2}^{n+1/2} + D_x|_{i,j+1/2}^{n-1/2}}{2} = \\ &\epsilon \left(\kappa_z \frac{E_x|_{i,j+1/2}^{n+1/2} - E_x|_{i,j+1/2}^{n-1/2}}{\Delta t} + \frac{\sigma_z}{\epsilon} \frac{E_x|_{i,j+1/2}^{n+1/2} + E_x|_{i,j+1/2}^{n-1/2}}{2} \right) , \end{aligned} \tag{24}$$

$$\begin{aligned} &\kappa_y \frac{D_y|_{i+1/2,j}^{n+1/2} - D_y|_{i+1/2,j}^{n-1/2}}{\Delta t} + \frac{\sigma_y}{\epsilon} \frac{D_y|_{i+1/2,j}^{n+1/2} + D_y|_{i+1/2,j}^{n-1/2}}{2} = \\ &\epsilon \left(\kappa_x \frac{E_y|_{i+1/2,j}^{n+1/2} - E_y|_{i+1/2,j}^{n-1/2}}{\Delta t} + \frac{\sigma_x}{\epsilon} \frac{E_y|_{i+1/2,j}^{n+1/2} + E_y|_{i+1/2,j}^{n-1/2}}{2} \right) , \end{aligned} \tag{25}$$

$$\begin{aligned} &\kappa_z \frac{B_z|_{i,j}^{n+1} - B_z|_{i,j}^{n}}{\Delta t} + \frac{\sigma_z}{\epsilon} \frac{B_z|_{i,j}^{n+1} + B_z|_{i,j}^{n}}{2} \\ &= \mu \left(\kappa_y \frac{H_z|_{i,j}^{n+1} - H_z|_{i,j}^{n}}{\Delta t} + \frac{\sigma_y}{\epsilon} \frac{H_z|_{i,j}^{n+1} + H_z|_{i,j}^{n}}{2} \right) . \end{aligned} \tag{26}$$

Isolando adequadamente os termos das equações (18-20) e (24-26),

$$\left(\frac{\kappa_y}{\Delta t} + \frac{\sigma_y}{2\epsilon} \right) D_x|_{i,j+1/2}^{n+1/2} = \left(\frac{\kappa_y}{\Delta t} - \frac{\sigma_y}{2\epsilon} \right) D_x|_{i,j+1/2}^{n-1/2} + \frac{H_z|_{i,j+1}^{n} - H_z|_{i,j}^{n}}{\Delta y} \tag{27}$$

$$\left(\frac{\kappa_z}{\Delta t} + \frac{\sigma_z}{2\epsilon} \right) D_y|_{i+1/2,j}^{n+1/2} = \left(\frac{\kappa_z}{\Delta t} - \frac{\sigma_z}{2\epsilon} \right) D_y|_{i+1/2,j}^{n-1/2} - \frac{H_z|_{i+1,j}^{n} - H_z|_{i,j}^{n}}{\Delta x} \tag{28}$$

$$\left(\frac{\kappa_x}{\Delta t} + \frac{\sigma_x}{2\epsilon} \right) B_z|_{i,j}^{n+1} = \left(\frac{\kappa_x}{\Delta t} - \frac{\sigma_x}{2\epsilon} \right) B_z|_{i,j}^{n} + \frac{E_y|_{i+1/2,j}^{n+1/2} - E_y|_{i-1/2,j}^{n+1/2}}{\Delta x} - \frac{E_x|_{i,j+1/2}^{n+1/2} - E_x|_{i,j-1/2}^{n+1/2}}{\Delta y} \tag{29}$$

$$\left(\frac{\kappa_z}{\Delta t} + \frac{\sigma_z}{2\epsilon} \right) E_x|_{i,j+1/2}^{n+1/2} = \left(\frac{\kappa_z}{\Delta t} - \frac{\sigma_z}{2\epsilon} \right) E_x|_{i,j+1/2}^{n-1/2} + \frac{1}{\epsilon} \left(\frac{\kappa_x}{\Delta t} + \frac{\sigma_x}{2\epsilon} \right) D_x|_{i,j+1/2}^{n+1/2} + \frac{1}{\epsilon} \left(\frac{\kappa_x}{\Delta t} - \frac{\sigma_x}{2\epsilon} \right) D_x|_{i,j+1/2}^{n-1/2} \tag{30}$$

$$\left(\frac{\kappa_x}{\Delta t} + \frac{\sigma_x}{2\epsilon} \right) E_y|_{i+1/2,j}^{n+1/2} = \left(\frac{\kappa_x}{\Delta t} - \frac{\sigma_x}{2\epsilon} \right) E_y|_{i+1/2,j}^{n-1/2} + \frac{1}{\epsilon} \left(\frac{\kappa_y}{\Delta t} + \frac{\sigma_y}{2\epsilon} \right) D_y|_{i+1/2,j}^{n+1/2} - \frac{1}{\epsilon} \left(\frac{\kappa_y}{\Delta t} - \frac{\sigma_y}{2\epsilon} \right) D_y|_{i+1/2,j}^{n-1/2} \tag{31}$$

$$\left(\frac{\kappa_y}{\Delta t} + \frac{\sigma_y}{2\epsilon} \right) H_z|_{i,j}^{n+1} = \left(\frac{\kappa_y}{\Delta t} - \frac{\sigma_y}{2\epsilon} \right) H_z|_{i,j}^{n} + \frac{1}{\mu} \left(\frac{\kappa_z}{\Delta t} + \frac{\sigma_z}{2\epsilon} \right) B_z|_{i,j}^{n+1} - \frac{1}{\mu} \left(\frac{\kappa_z}{\Delta t} - \frac{\sigma_z}{2\epsilon} \right) B_z|_{i,j}^{n} . \tag{32}$$

3 Caso 3-d

Os elementos da equação 11 12 são

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \kappa_y \frac{\partial D_x}{\partial t} + \frac{\sigma_y}{\epsilon} \partial D_x \tag{33}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \kappa_z \frac{\partial D_y}{\partial t} + \frac{\sigma_z}{\epsilon} \partial D_y \tag{34}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \kappa_x \frac{\partial D_z}{\partial t} + \frac{\sigma_x}{\epsilon} \partial D_z \tag{35}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\kappa_y \frac{\partial B_x}{\partial t} - \frac{\sigma_y}{\epsilon} \partial B_x \tag{36}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\kappa_z \frac{\partial B_y}{\partial t} - \frac{\sigma_z}{\epsilon} \partial B_y \tag{37}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\kappa_x \frac{\partial B_z}{\partial t} - \frac{\sigma_x}{\epsilon} \partial B_z \tag{38}$$

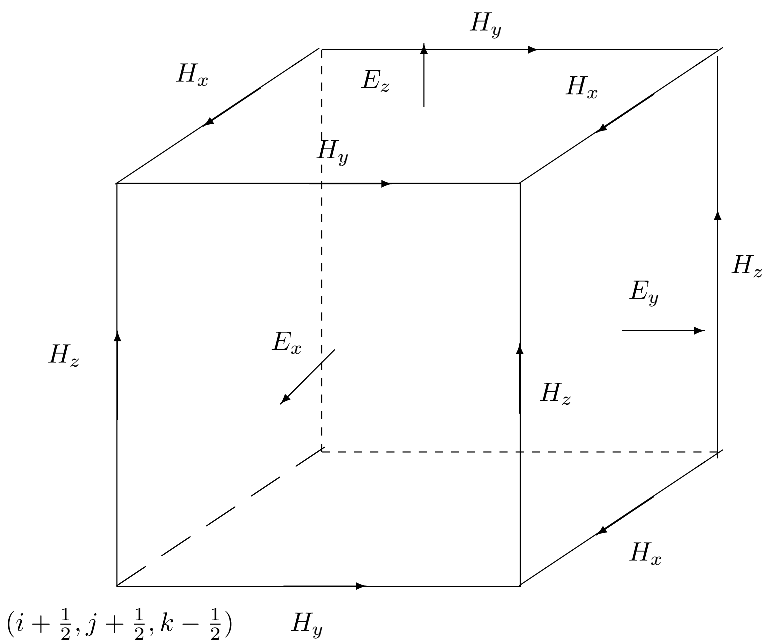

Figura 1: Célula de Yee

Discretizamos as equações de (33-38) usando as posições da célula de Yee. Os passos de tem são discretizados usando o algoritmo *leap-frog*:

$$\frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2} - H_z|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y|_{i+1/2,j,k+1/2}^{n+1/2} - H_y|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z} = \frac{\kappa_y}{\Delta t}\left(D_x|_{i+1/2,j,k}^{n+1} - D_x|_{i+1/2,j,k}^{n}\right) + \frac{\sigma_y}{2\epsilon}\left(D_x|_{i+1/2,j,k}^{n+1} + D_x|_{i+1/2,j,k}^{n}\right), \quad (39)$$

$$\frac{H_x|_{i,j+1/2,k+1/2}^{n+1/2} - H_x|_{i,j+1/2,k-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2} - H_z|_{i-1/2,j+1/2,k}^{n+1/2}}{\Delta x} = \frac{\kappa_z}{\Delta t}\left(D_y|_{i,j+1/2,k}^{n+1} - D_y|_{i,j+1/2,k}^{n}\right) + \frac{\sigma_z}{2\epsilon}\left(D_y|_{i,j+1/2,k}^{n+1} + D_y|_{i,j+1/2,k}^{n}\right), \quad (40)$$

$$\frac{H_y|_{i+1/2,j,k+1/2}^{n+1/2} - H_y|_{i-1/2,j,k+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{i,j+1/2,k+1/2}^{n+1/2} - H_x|_{i,j-1/2,k+1/2}^{n+1/2}}{\Delta y} = \frac{\kappa_x}{\Delta t}\left(D_z|_{i,j,k+1/2}^{n+1} - D_z|_{i,j,k+1/2}^{n}\right) + \frac{\sigma_x}{2\epsilon}\left(D_z|_{i,j,k+1/2}^{n+1} + D_z|_{i,j,k+1/2}^{n}\right), \quad (41)$$

$$\frac{E_z|_{i,j,k+1/2}^{n+1} - E_z|_{i,j-1,k+1/2}^{n+1}}{\Delta y} - \frac{E_y|_{i,j+1/2,k}^{n+1} - E_y|_{i,j+1/2,k-1}^{n+1}}{\Delta z} = -\frac{\kappa_y}{\Delta t}\left(B_x|_{i,j+1/2,k+1/2}^{n+3/2} - B_x|_{i,j+1/2,k+1/2}^{n+1/2}\right) - \frac{\sigma_y}{2\epsilon}\left(B_x|_{i,j+1/2,k+1/2}^{n+3/2} + B_x|_{i,j+1/2,k+1/2}^{n+1/2}\right), \quad (42)$$

$$\frac{E_x|_{i+1/2,j,k}^{n+1} - E_x|_{i+1/2,j,k-1}^{n+1}}{\Delta z} - \frac{E_z|_{i,j,k+1/2}^{n+1} - E_y|_{i-1,j,k+1/2}^{n+1}}{\Delta x} = -\frac{\kappa_z}{\Delta t}\left(B_y|_{i+1/2,j,k+1/2}^{n+3/2} - B_y|_{i+1/2,j,k+1/2}^{n+1/2}\right) - \frac{\sigma_z}{2\epsilon}\left(B_y|_{i+1/2,j,k+1/2}^{n+3/2} + B_y|_{i+1/2,j,k+1/2}^{n+1/2}\right), \quad (43)$$

$$\frac{E_y|_{i,j+1/2,k}^{n+1} - E_y|_{i-1,j+1/2,k}^{n+1}}{\Delta x} - \frac{E_x|_{i+1/2,j,k}^{n+1} - E_y|_{i+1/2,j-1,k}^{n+1}}{\Delta y} = -\frac{\kappa_x}{\Delta t}\left(B_z|_{i+1/2,j+1/2,k}^{n+3/2} - B_z|_{i+1/2,j+1/2,k}^{n+1/2}\right) - \frac{\sigma_x}{2\epsilon}\left(B_z|_{i+1/2,j+1/2,k}^{n+3/2} + B_z|_{i+1/2,j+1/2,k}^{n+1/2}\right). \quad (44)$$

Isolando adequadamente os temos:

$$\alpha_y D_x|_{i+1/2,j,k}^{n+1} = \beta_y D_x|_{i+1/2,j,k}^{n} + \frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2} - H_z|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y|_{i+1/2,j,k+1/2}^{n+1/2} - H_y|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}, \quad (45)$$

$$\alpha_z D_y|_{i,j+1/2,k}^{n+1} = \beta_z D_y|_{i,j+1/2,k}^{n} + \frac{H_x|_{i,j+1/2,k+1/2}^{n+1/2} - H_x|_{i,j+1/2,k-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2} - H_z|_{i-1/2,j+1/2,k}^{n+1/2}}{\Delta x}, \quad (46)$$

$$\alpha_x D_z|^{n+1}_{i,j,k+1/2} = \beta_x D_z|^{n}_{i,j,k+1/2} + \frac{H_y|^{n+1/2}_{i+1/2,j,k+1/2} - H_y|^{n+1/2}_{i-1/2,j,k+1/2}}{\Delta x} - \frac{H_x|^{n+1/2}_{i,j+1/2,k+1/2} - H_x|^{n+1/2}_{i,j-1/2,k+1/2}}{\Delta y} \,, \tag{47}$$

$$\alpha_y B_x|^{n+3/2}_{i,j+1/2,k+1/2} = \beta_y B_x|^{n+1/2}_{i,j+1/2,k+1/2} + \frac{E_y|^{n+1}_{i,j+1/2,k} - E_y|^{n+1}_{i,j+1/2,k-1}}{\Delta z} - \frac{E_z|^{n+1}_{i,j,k+1/2} - E_z|^{n+1}_{i,j-1,k+1/2}}{\Delta y} \,, \tag{48}$$

$$\alpha_z B_y|^{n+3/2}_{i+1/2,j,k+1/2} = \beta_z B_y|^{n+1/2}_{i+1/2,j,k+1/2} + \frac{E_z|^{n+1}_{i,j,k+1/2} - E_z|^{n+1}_{i-1,j,k+1/2}}{\Delta x} - \frac{E_x|^{n+1}_{i+1/2,j,k} - E_x|^{n+1}_{i+1/2,j,k-1}}{\Delta z} \,, \tag{49}$$

$$\alpha_x B_z|^{n+3/2}_{i+1/2,j+1/2,k} = \beta_x B_z|^{n+1/2}_{i+1/2,j+1/2,k} + \frac{E_x|^{n+1}_{i+1/2,j,k} - E_x|^{n+1}_{i+1/2,j-1,k}}{\Delta y} - \frac{E_y|^{n+1}_{i,j+1/2,k} - E_y|^{n+1}_{i-1,j+1/2,k}}{\Delta x} \,, \tag{50}$$

onde,

$$\alpha_{x,y,z} = \frac{\kappa_{x,y,z}}{\Delta x, y, z} + \frac{\sigma_{x,y,z}}{2\epsilon} \,, \tag{51}$$

$$\beta_{x,y,z} = \frac{\kappa_{x,y,z}}{\Delta x, y, z} + \frac{\sigma_{x,y,z}}{2\epsilon} \,. \tag{52}$$

Precisamos também da discretização das relações constitutivas:

$$\alpha_z E_x|^{n+1/2}_{i+1/2,j,k} = \beta_z E_x|^{n-1/2}_{i+1/2,j,k} + \frac{1}{\epsilon}\alpha_x D_x|^{n+1/2}_{i+1/2,j,k} - \frac{1}{\epsilon}\beta_x D_x|^{n-1/2}_{i+1/2,j,k} \tag{53}$$

$$\alpha_x E_y|^{n+1/2}_{i,j+1/2,k} = \beta_x E_y|^{n-1/2}_{i,j+1/2,k} + \frac{1}{\epsilon}\alpha_y D_y|^{n+1/2}_{i,j+1/2,k} - \frac{1}{\epsilon}\beta_y D_y|^{n-1/2}_{i,j+1/2,k} \tag{54}$$

$$\alpha_y E_z|^{n+1/2}_{i,j,k+1/2} = \beta_y E_z|^{n-1/2}_{i,j,k+1/2} + \frac{1}{\epsilon}\alpha_z D_z|^{n+1/2}_{i,j,k+1/2} - \frac{1}{\epsilon}\beta_z D_z|^{n-1/2}_{i,j,k+1/2} \tag{55}$$

$$\alpha_z H_x|^{n+1}_{i,j+1/2,k+1/2} = \beta_z H_x|^{n}_{i,j+1/2,k+1/2} + \frac{1}{\mu}\alpha_x B_x|^{n+1}_{i,j+1/2,k+1/2} - \frac{1}{\mu}\beta_x B_x|^{n}_{i,j+1/2,k+1/2} \,. \tag{56}$$

$$\alpha_x H_y|^{n+1}_{i+1/2,j,k+1/2} = \beta_x H_y|^{n}_{i+1/2,j,k+1/2} + \frac{1}{\mu}\alpha_y B_y|^{n+1}_{i+1/2,j,k+1/2} - \frac{1}{\mu}\beta_y B_y|^{n}_{i+1/2,j,k+1/2} \,. \tag{57}$$

$$\alpha_y H_z|^{n+1}_{i+1/2,j+1/2,k} = \beta_y H_z|^{n}_{i+1/2,j+1/2,k} + \frac{1}{\mu}\alpha_z B_z|^{n+1}_{i+1/2,j+1/2,k} - \frac{1}{\mu}\beta_z B_z|^{n}_{i+1/2,j+1/2,k} \,. \tag{58}$$