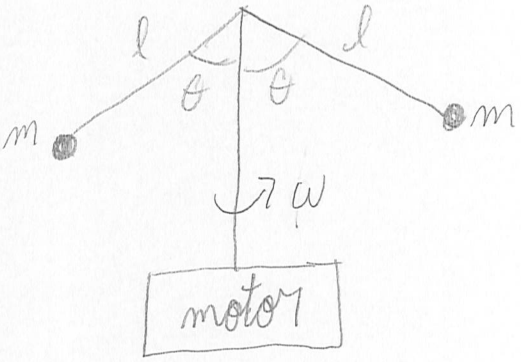
Exemplos de Hamiltoniano

Exemplo 1 – Brinquedo rotatório de hastes rígidas



$$V = -2mgl\cos\theta$$

$$x = l\sin\theta\cos(\omega t)$$
$$y = l\sin\theta\sin(\omega t)$$
$$z = l\cos\theta$$

Função do motor:
manter ω constante

$$\dot{x} = l\dot{\theta}\cos\theta\cos(\omega t) - l\omega\sin\theta\sin(\omega t)$$
$$\dot{y} = l\dot{\theta}\cos\theta\sin(\omega t) + l\omega\sin\theta\cos(\omega t)$$
$$\dot{z} = l\dot{\theta}\sin\theta$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2\dot{\theta}^2\cos^2\theta + l^2\omega^2\sin^2\theta + l^2\dot{\theta}^2\sin^2\theta$$
$$= l^2\dot{\theta}^2 + l^2\omega^2\sin^2\theta$$

$$T = 2 \times \frac{mv^2}{2} = ml^2(\dot{\theta}^2 + \omega^2\sin^2\theta)$$

$$\boxed{L = T - V = ml^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + 2mgl\cos\theta}$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = 2ml^2\dot{\theta}$$

$$\dot{\theta} = \frac{p_\theta}{2ml^2}$$

$$H = p_\theta\dot{\theta} - L$$
$$= \frac{p_\theta^2}{2ml^2} - ml^2\omega^2\sin^2\theta$$
$$-2mgl\cos\theta$$

Observe que o Hamiltoniano não é igual à energia $E = T + V$.
A equivalência Hamiltoniano-energia é válida em sistemas que conservam energia.

$$\dot p_\theta = -\frac{\partial H}{\partial \theta}$$

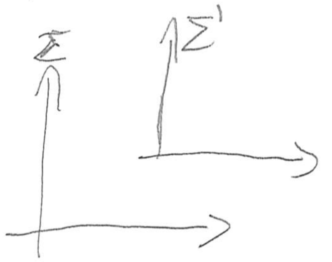
$$= -\left[-ml^2\omega^2 \sin 2\theta + 2mgl\sin\theta\right]$$

$$\boxed{\dot p_\theta = ml\left(l\omega^2 \sin 2\theta - g\sin\theta\right)}$$

$$\boxed{\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}}$$

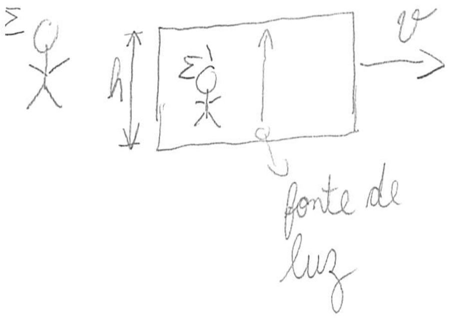
Exemplo 2 – Relatividade restrita

Relatividade Newtoniana

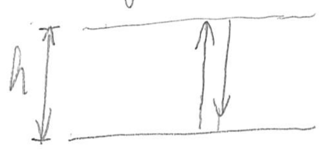


$dt = dt'$ Tempo flui uniformemente e de forma igual em todos os referenciais inerciais

Experimento de Michelson-Morley: velocidade da luz medida é a mesma em todos os referenciais inerciais

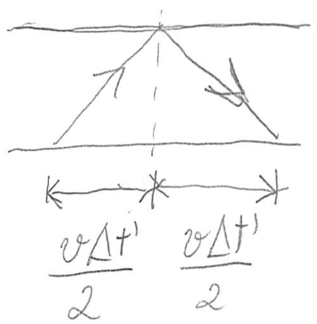


Referencial Σ'



$\Delta t' = \frac{2h}{c}$: tempo de ida e volta da luz

Referencial Σ



$$\left(\frac{c\Delta t}{2}\right)^2 = h^2 + \left(\frac{v\Delta t}{2}\right)^2$$

$$(c^2 - v^2)\Delta t^2 = (2h)^2$$

$$(\Delta t)^2 = \frac{(2h)^2}{c^2 - v^2} = \frac{(2h)^2}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

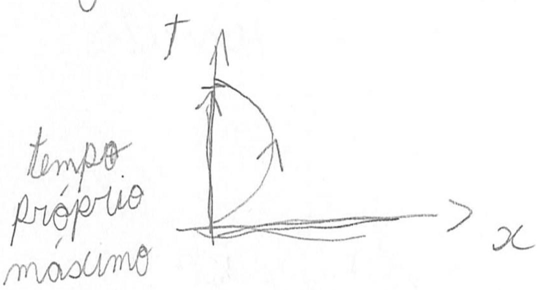
$$\boxed{\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Note: $\Delta t'$ não depende de velocidade alguma

$$(\Delta t')^2 = (\Delta t)^2\left(1 - \frac{v^2}{c^2}\right) = \Delta t^2 - \frac{\Delta x^2}{c^2}$$

$$(d\tau)^2 = dt^2 - \frac{dx^2}{c^2}$$ ∴ diferencial do tempo próprio, que é o mesmo para todos os referenciais inerciais

Inércia em relatividade restrita: tendência em maximizar tempo próprio



$$\tau = \int dt \sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

Considere a Lagrangiana $L = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}}$

$$S = -mc^2 \int dt \sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

Momento linear

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}$$

Hamiltoniano

$$H = p_x \dot{x} - L$$

$$= \frac{m\dot{x}^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} + mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

$$= \frac{mc^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}$$

$$E = \frac{mc^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}$$

Se $\dot{x} << c \implies E \simeq mc^2 + \frac{m\dot{x}^2}{2}$

Se $\dot{x} = 0$, $E = mc^2$

$$\gamma = \frac{1}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} \Rightarrow \begin{array}{l} E = \gamma m c^2 \\ p_x = \gamma m \dot{x} \end{array}$$

$$E^2 - p_x^2 c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{\dot{x}^2}{c^2}\right) = m^2 c^4$$

$$E = \sqrt{m^2 c^4 + p^2 c^2}$$

$ds^2 = c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$: relatividade restrita

$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu$, $dx^0 = dt$

$g_{\mu\nu}$: métrica do espaço tempo