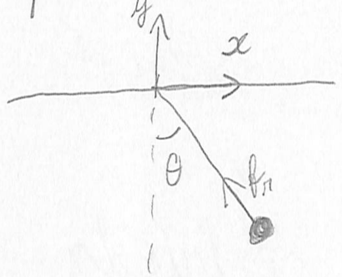
Lagrangianas com vínculos

s vínculos : $\displaystyle\sum_{k=1}^{n} a_{lk}\,\dot{q}_k + a_{lt} = 0, \quad l = 1, \dots, s$

n Eqs. de Euler-Lagrange : $\displaystyle\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{s} \lambda_l a_{lk}, \quad k = 1, \dots, n$

Encontrar $q_k(t)$ e $\lambda_l(q, \dot{q}, t)$.

Exemplo 1: pêndulo simples



$x = r\sin\theta$

$y = -r\cos\theta$

$\dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$\dot{y} = -\dot{r}\cos\theta + r\dot{\theta}\sin\theta$

$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$

A Lagrangiana sem vínculo é calculada sabendo-se T e V.

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \Rightarrow \boxed{T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)}$

$V = mgy \Rightarrow \boxed{V = -mgr\cos\theta}$

$$\boxed{L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta}$$

2 coordenadas generalizadas $q_1 = r$, $q_2 = \theta$

1 vínculo $f(r) = r - l = 0 \Rightarrow \dot{r} = 0$

$\boxed{a_{11} = 1, \; a_{12} = 0, \; a_{1t} = 0}$

$r\ (k=1)$: $\displaystyle\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda a_{11} = \lambda = m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta$

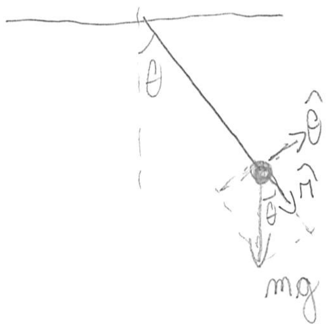
$\theta\ (k=2)$: $\displaystyle\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda a_{12} = 0 = \frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta$

$= 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta$

$$\begin{cases} m\ddot{r} - m r \dot{\theta}^2 - m g \cos\theta = \lambda \\ \ddot{\theta} + 2\dfrac{\dot{r}}{r}\dot{\theta} + \dfrac{g}{r}\sin\theta = 0 \\ r = l \Rightarrow \dot{r} = 0,\ \ddot{r} = 0 \end{cases} \Rightarrow \begin{cases} \lambda = -\dfrac{m}{l}(l\dot{\theta})^2 - m g \cos\theta \\ \ddot{\theta} + \dfrac{g}{l}\sin\theta = 0 \end{cases}$$

Interpretação do multiplicador de Lagrange

$$\vec{P} = m g \hat{y} = m g \cos\theta\, \hat{r} - m g \sin\theta\, \hat{\theta}$$

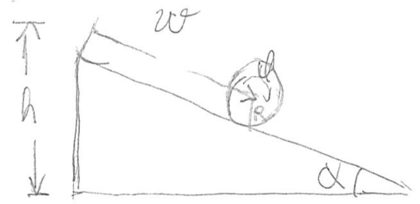


Força centrípeta

$$\vec{F}_c = -\frac{m v^2}{r}\hat{r} = -\frac{m}{l}(l\dot{\theta})^2$$

λ: Força na direção $\hat{r}$ que gera a força centrípeta correta e compensa o peso (tração, força de vínculo)

Exemplo 2: Cilindro rolando no plano inclinado



Condição de rolar sem deslizar (1 vínculo)

$$R\dot{\varphi} = \dot{w}$$

Em termos diferenciais

$$R\, d\varphi - dw = 0 \Rightarrow \begin{aligned} a_{l\varphi} &= R \\ a_{lw} &= -1 \end{aligned}$$

Em outra seção do curso, demonstramos que

$$\boxed{\begin{aligned} T &= T_{cm} + T_{rot} \\ &= \frac{1}{2} m \dot{w}^2 + \frac{1}{2} I \dot{\varphi}^2 \end{aligned}} \qquad \left(I = \frac{1}{2} m R^2\right)$$

Por outro lado

$$\boxed{V = mgh - mgw\sin\alpha}$$

A Lagrangiana é dada por

$$\boxed{\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{w}^2 + \frac{1}{2}I\dot{\varphi}^2 + mgw\sin\alpha - mgh \end{aligned}}$$

Precisamos de apenas uma força de vínculo λ

Eqs. Euler- Lagrange

$$\varphi: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = \lambda a_{l\varphi} = R\lambda$$

$$\boxed{I\ddot{\varphi} - \lambda R = 0}$$

$$w: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}}\right) - \frac{\partial L}{\partial w} = \lambda a_{lw} = -\lambda$$

$$(*) \quad \boxed{m\ddot{w} - mg\sin\alpha + \lambda = 0}$$

$$R\dot{\varphi} = \dot{w} \Rightarrow \boxed{R\ddot{\varphi} = \ddot{w}}$$

Assim: $I\ddot{\varphi} - \lambda R = \left(\frac{1}{2}mR^2\right)\left(\frac{\ddot{w}}{R}\right) - \lambda R = 0$

$$\boxed{m\ddot{w} = 2\lambda}$$

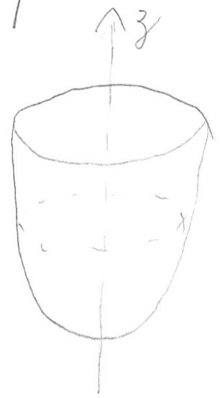
Substituindo em $(*)$ $\quad 3\lambda - mg\sin\alpha = 0$

$$\boxed{\lambda = \frac{mg}{3}\sin\alpha}$$

λ: força de atrito que torna a condição de rolar sem deslizar possível

Novamente em $(*)$: $\ddot{w} = \frac{2}{3}mg\sin\alpha$

Exemplo 3: Partícula se movendo em paraboloide



Paraboloide satisfaz

$$x^2 + y^2 = \alpha z$$

Determinar a Lagrangiana e a equação de movimento com força de vínculo

$$x = r\cos\theta \qquad\qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$y = r\sin\theta \quad \Longrightarrow \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$z = z \qquad\qquad \dot{z} = \dot{z}$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$$

Lagrangiana $\quad T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

$$V = mgz$$

$$\boxed{L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz}$$

$$f(r, z) = r^2 - az = 0$$

$$\boxed{2r\,dr - \alpha\,dz = 0} \qquad a_{1r} = 2r, \quad a_{1z} = -\alpha, \quad a_{1\theta} = 0$$

r: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda a_{1r} = 2r\lambda$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 = 2r\lambda}$$

z: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \lambda a_{1z} = -\alpha$

$$\boxed{m\ddot{z} = -mg - \alpha\lambda}$$

θ: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}(mr^2\dot{\theta})$

$$\boxed{mr^2\dot{\theta} = \text{constante}}$$

$$2r\,dr - \alpha\,dz = 0$$

$$\Longrightarrow \boxed{2r\dot{r} - \alpha\dot{z} = 0}$$

A fim de ganhar algum entendimento do problema vamos considerar a situação ideal $z=$ constante

$\dot{z}=0 \Rightarrow \dot{r}=0 \Rightarrow r=$ constante, $\ddot{r}=0$

$$\boxed{\lambda = -\frac{mg}{\alpha}}$$

$$m r \dot{\theta}^2 = -2r\lambda$$

$$\dot{\theta}^2 = -\frac{2}{m}\lambda = \frac{2g}{\alpha}$$

$\dot{\theta} = \sqrt{\frac{2g}{\alpha}}$: velocidade angular a ser imposta para que tal movimento ocorra