Gradiente em coordenadas esféricos

[Retirado de: "Quantum Mechanics, concepts and applications"]

em retangulares temos

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Agora,

$$\hat{x} = \hat{r}\,\mathrm{sen}\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\,\mathrm{sen}\varphi$$

$$\hat{y} = \hat{r}\,\mathrm{sen}\theta\,\mathrm{sen}\varphi + \hat{\theta}\cos\theta\,\mathrm{sen}\varphi + \hat{\varphi}\cos\varphi$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\,\mathrm{sen}\theta$$

pois

$$\begin{cases} x = r\,\mathrm{sen}\theta\cos\varphi \\ y = r\,\mathrm{sen}\theta\,\mathrm{sen}\varphi \\ z = r\cos\theta \end{cases}$$

Assim temos que

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x}$$

e assim por diante para y e z

Agora:

$$dx = dr\,\mathrm{sen}\theta\cos\varphi + r\cos\theta\cos\varphi\,d\theta - r\,\mathrm{sen}\theta\,\mathrm{sen}\varphi\,d\varphi$$

$$dy = dr\,\mathrm{sen}\theta\,\mathrm{sen}\varphi + r\cos\theta\,\mathrm{sen}\varphi\,d\theta + r\,\mathrm{sen}\theta\cos\varphi\,d\varphi$$

$$dz = dr\cos\theta - r\,\mathrm{sen}\theta\,d\theta$$

O que resulta em:

$$dr = \operatorname{sen}\theta\cos\varphi\, dx + \operatorname{sen}\theta\operatorname{sen}\varphi\, dy + \cos\theta\, dz$$

$$d\theta = \frac{1}{r}\cos\theta\cos\varphi\, dx + \frac{1}{r}\cos\theta\operatorname{sen}\varphi\, dy - \frac{1}{r}\operatorname{sen}\theta\, dz$$

$$d\varphi = -\frac{\operatorname{sen}\varphi}{r\operatorname{sen}\theta}\, dx + \frac{\cos\varphi}{r\operatorname{sen}\theta}\, dy$$

$\Rightarrow$ temos, por exemplo: $\boxed{\dfrac{\partial r}{\partial x} = \operatorname{sen}\theta\cos\varphi}$ $\boxed{\dfrac{\partial \theta}{\partial y} = \dfrac{1}{r}\cos\theta\operatorname{sen}\varphi}$

e assim por diante para os demais termos

$$\therefore\ \vec{\nabla} = \underbrace{\frac{\partial}{\partial x}\hat{x}}_{①} + \underbrace{\frac{\partial}{\partial y}\hat{y}}_{②} + \underbrace{\frac{\partial}{\partial z}\hat{z}}_{③}$$

$$\frac{\partial}{\partial x}\hat{x} = \left[\frac{\partial}{\partial r}\cdot(\operatorname{sen}\theta\cos\varphi) + \left(\frac{1}{r}\cos\theta\cos\varphi\right)\frac{\partial}{\partial\theta} + \left(-\frac{\operatorname{sen}\varphi}{r\operatorname{sen}\theta}\right)\frac{\partial}{\partial\varphi}\right]\times$$

$$\times\left[\hat{r}\operatorname{sen}\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\operatorname{sen}\varphi\right]$$

$$\frac{\partial}{\partial y}\hat{y} = \left[\operatorname{sen}\theta\operatorname{sen}\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\operatorname{sen}\varphi\frac{\partial}{\partial\theta} + \left(\frac{\cos\varphi}{r\operatorname{sen}\theta}\right)\frac{\partial}{\partial\varphi}\right]\times$$

$$\times\left[\hat{r}\operatorname{sen}\theta\operatorname{sen}\varphi + \hat{\theta}\cos\theta\operatorname{sen}\varphi + \hat{\varphi}\cos\varphi\right.$$

$$\frac{\partial}{\partial z}\hat{z} = \left[\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\operatorname{sen}\theta\frac{\partial}{\partial\theta}\right]\times\left(\hat{r}\cos\theta - \hat{\theta}\operatorname{sen}\theta\right)$$

Com isso, isolamos cada termo ($\hat{r}, \hat{\theta}$ e $\hat{\varphi}$)

p/ $\hat{r}$:

$$\hat{r}\cdot\left(\operatorname{sen}^2\theta\cos^2\varphi\frac{\partial}{\partial r}+\frac{1}{r}\operatorname{sen}\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta}+\frac{1}{r}(-\operatorname{sen}\varphi\cos\varphi)\frac{\partial}{\partial\varphi}\right)+$$

$$+\hat{r}\cdot\left(\operatorname{sen}^2\theta\operatorname{sen}^2\varphi\frac{\partial}{\partial r}+\frac{1}{r}\operatorname{sen}\theta\cos\theta\operatorname{sen}^2\varphi\frac{\partial}{\partial\theta}+\frac{1}{r}(\operatorname{sen}\varphi\cos\varphi)\frac{\partial}{\partial\varphi}\right)+$$

$$+\hat{r}\cdot\left(\cos^2\theta\frac{\partial}{\partial r}-\frac{1}{r}\operatorname{sen}\theta\cos\theta\frac{\partial}{\partial\theta}\right)=$$

$$=\hat{r}\cdot\left[\frac{\partial}{\partial r}+\frac{1}{r}\cdot\xrightarrow{\text{zero}}\cancel{(\operatorname{sen}\theta\cos\theta-\operatorname{sen}\theta\cos\theta)}\frac{\partial}{\partial\theta}+\frac{1}{r}\xrightarrow{\text{zero}}\cancel{(-\operatorname{sen}\varphi\cos\varphi+\operatorname{sen}\varphi\cos\varphi)}\frac{\partial}{\partial\varphi}\right]$$

$\therefore$ termo em $\hat{r}$: $\boxed{\vec{\nabla}_r=\frac{\partial}{\partial r}\hat{r}}$

em $\hat{\theta}$:

$$\hat{\theta}\cdot\left[\cos\theta\cos^2\varphi\operatorname{sen}\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\cos\theta}{\operatorname{sen}\theta}\cos\varphi\operatorname{sen}\varphi\frac{\partial}{\partial\varphi}\right]+$$

$$+\hat{\theta}\cdot\left[\cos\theta\operatorname{sen}\theta\operatorname{sen}^2\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos^2\theta\operatorname{sen}^2\varphi\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\operatorname{sen}\theta}\cos\varphi\operatorname{sen}\varphi\frac{\partial}{\partial\varphi}\right]+$$

$$+\hat{\theta}\cdot\left\{\operatorname{sen}\theta\cos\theta\frac{\partial}{\partial r}+\frac{1}{r}\operatorname{sen}^2\theta\frac{\partial}{\partial\theta}\right]$$

$$\Rightarrow=\hat{\theta}\cdot\left[\frac{1}{r}\frac{\partial}{\partial\theta}\right]\quad\therefore\quad\vec{\nabla}_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}$$

Por fim, para $\hat{\varphi}$:

$$\hat{\varphi} \cdot \left[-\operatorname{sen}\varphi \operatorname{sen}\theta \cos\varphi \frac{\partial}{\partial r} - \frac{1}{r} \operatorname{sen}\varphi \cos\varphi \cos\theta \frac{\partial}{\partial\theta} + \frac{\operatorname{sen}^2\varphi}{r\operatorname{sen}\theta} \frac{\partial}{\partial\varphi} \right] +$$

$$\hat{\varphi} \cdot \left[\cos\varphi \operatorname{sen}\varphi \operatorname{sen}\theta \frac{\partial}{\partial r} + \frac{1}{r} \cos\varphi \cos\theta \operatorname{sen}\varphi \frac{\partial}{\partial\theta} + \frac{\cos^2\varphi}{r\operatorname{sen}\theta} \frac{\partial}{\partial\varphi} \right]$$

$$= \hat{\varphi} \cdot \left(\frac{1}{r\operatorname{sen}\theta} \right) \frac{\partial}{\partial\varphi} \Rightarrow \vec{\nabla}_\varphi = \frac{1}{r\operatorname{sen}\theta} \frac{\partial}{\partial\varphi} \hat{\varphi}$$

Assim, em coordenadas esféricas:

$$\boxed{\vec{\nabla} = \left(\frac{\partial}{\partial r} \right) \hat{r} + \left(\frac{1}{r} \frac{\partial}{\partial\theta} \right) \hat{\theta} + \left(\frac{1}{r\operatorname{sen}\theta} \frac{\partial}{\partial\varphi} \right) \hat{\varphi}}$$