## Constituent Gluons in Static Quark Systems

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## Gluons as constituent particles?

Perturbative degrees of freedom: quarks and gluons.
But do particle-like gluons make any sense as constituents of hadronic bound states, particularly highly excited bound states?



Regge trajectories $\Longrightarrow$ "spinning stick," or string, or flux tube between the quark and antiquark.

Does this line-like object have a point-like (constituent gluon) substructure?

## Gluon chain model

An old idea: The gluon chain model (Thorn and JG, 2001)



Representation in Coulomb gauge:

$$
|n\rangle_{\bar{q} q}=\int \prod_{i=1}^{n} d^{3} x_{i} \Psi_{k_{1} \ldots k_{n}}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \bar{q}^{\dagger}(0) A_{k_{1}}\left(x_{1}\right) A_{k_{2}}\left(x_{2}\right) \ldots A_{k_{n}}\left(x_{n}\right) q^{\dagger}(R)|0\rangle_{\mathrm{vac}}
$$

## Gluon chain model, continued

In its original incarnation it was thought that the Coulomb potential between static sources would eventually rise faster than linear.

This turns out not to be true...the Coulomb potential is asymptotically linear.
(Olejnik and JG, 2003)
In that case, there seems to be no energetic advantage to adding gluons. The energy expectation value can only go up.

However, the Coulomb string tension is far larger than the gauge-invariant asymptotic string tension. It can't be the whole story.

## (Over)Confining Coulomb potential

The instantaneous Coulomb potential is derived from the energy VEV of a $q \bar{q}$ state

$$
|0\rangle_{\bar{q} q}=\bar{q}_{i}^{\dagger}(0) q_{i}^{\dagger}(R)|0\rangle_{\mathrm{vac}}
$$

where $|0\rangle_{\text {vac }}$ is the non-perturbative vacuum state. There are no constituent gluons. Then

$$
\begin{aligned}
V(R) & =\bar{q} \alpha\langle 0| H|0\rangle_{\bar{q} q} \\
& =-\lim _{t \rightarrow 0} \frac{d}{d t} \log \left\{\langle 0| \bar{q}(R) q(0) e^{-H t} q^{\dagger}(0) \bar{q}^{\dagger}(R)|0\rangle\right\} \\
& =-\lim _{t \rightarrow 0} \frac{d}{d t} \log \left\langle\operatorname{Tr}\left[L_{t}(\mathbf{0}) L_{t}^{\dagger}(\mathbf{R})\right]\right\rangle
\end{aligned}
$$

where

$$
L_{t}(\mathbf{x}) \equiv T \exp \left[i g \int_{0}^{t} d t A_{0}(\mathbf{x}, t)\right]
$$

is a Wilson line of length $t$ in the time direction.

## Coulomb potential II

On the lattice :

$$
\begin{aligned}
V(R) & =\lim _{\beta \rightarrow \infty}\left(\frac{V_{L}\left(R_{L}, \beta\right)}{a(\beta)}\right) \\
V_{L}(R, \beta) & =-\log \left\langle\frac{1}{3} \operatorname{Tr} U_{0}(\mathbf{0}, 0) U_{0}^{\dagger}(\mathbf{R}, 0)\right\rangle
\end{aligned}
$$

We get the Coulomb potential from the correlator of timelike links in Coulomb gauge.

## Coulomb potential III

From numerical simulations the Coulomb potential turns out to be linearly confining, and fits the form

$$
V_{\text {coul }}(R)=\sigma_{\text {coul }} R-\frac{\pi}{12} \frac{1}{R}
$$




The Coulomb string tension $\sigma_{\text {coul }}=(891 \mathrm{MeV})^{2}$ is about four times larger than the asymptotic string tension. Too much of a good thing!

In fitting the data, in physical units, to

$$
V(R)=\sigma R-\frac{\gamma}{R}+\frac{c}{a}
$$

we are able to isolate (and subtract) the self energy $c / a$, and find that the continuum limit of $\gamma$ seems to be Lüscher's value of $\pi / 12$.


This is actually a little puzzling: what does the Coulomb energy have to do with string theory?
But the greater puzzle is why the asymptotic string tension is a factor of four lower than the Coulomb string tension.

## A toy model

$V_{\text {coul }}(R)$ is the energy of a zero constituent-gluon state

$$
|0\rangle_{\bar{q} q}=\bar{q}_{i}^{\dagger}(0) q_{i}^{\dagger}(R)|0\rangle_{\mathrm{vac}}
$$

It seems like adding more gluons (gluon-chain model) would just increase the energy. However, $|0\rangle_{\bar{q} q}$ is not an eigenstate of $H$.

Let $|n\rangle_{\bar{q} q}$ denote a state with $n$ constituent gluons arranged more-or-less in a line between the quark and antiquark. The distance between gluons is $\approx R / n$, and we'll suppose fluctuations in the transverse direction are of order $1 / a$. Then the kinetic energy of each gluon is roughly

$$
\begin{equation*}
\mathrm{KE} \approx \sqrt{\frac{n^{2}}{R^{2}}+a^{2}} \tag{1}
\end{equation*}
$$

(a dynamical mass can be absorbed in the $a^{2}$ term). Adding up all the inter-gluon Coulomb interactions,

$$
\begin{equation*}
\text { total Coulomb energy } \approx \sigma_{c} R \tag{2}
\end{equation*}
$$

## toy model II

Then the energy of the $n$-gluon state is the diagonal element of the Hamiltonian

$$
H_{n n}=n \sqrt{\frac{n^{2}}{R^{2}}+a^{2}}+\sigma_{c} R
$$

We suppose, in this toy model, that states with different numbers of constituent gluons are orthogonal. But what is the effect of off-diagonal terms?
Let us suppose that these have the form

$$
H_{n, n+1}=H_{n+1, n}=(n+1) \alpha\left(\frac{R}{n+1}\right)^{p}
$$

Pick some parameters $a$, $\alpha$ and power dependence $p$, diagonalize $H$, and find the lowest energy eigenvalue. This is the interquark potential $V(R)$.

Can it be lower than $\sigma_{c} R$ ? Is it still linear?

## toy model III

The uppermost line is the Coulomb potential $\sigma_{c} R$, with units $\sigma_{c}=1$. All other lines have a smaller slope.


Remarkably, the off-diagonal elements lower the Coulomb string tension while preserving the linearity of the potential, almost irrespective of the $R$-dependence of the off-diagonal terms.

## toy model IV

One can also compute the excitation spectrum in this model, and the excitations are also linear in separation $R$,


The question is whether we can go beyond this toy model. Is this really what is going on?

## Lattice-improved tree diagrams

$H_{m n}$ is calculated much like S-matrix elements in renormalized perturbation theory.

$$
H_{m n}=-\lim _{t \rightarrow 0} \frac{d}{d t}\langle m| e^{-H t}|n\rangle
$$

## There are a finite number of tree diagrams contributing to each $H_{m n}$.

The building blocks are dressed propagators, and 1PI n-point vertex functions.
We propose to

- construct a finite basis
- keep all the tree diagrams
- obtain dressed propagators from the lattice
- truncate vertex functions
- diagonalize the Hamiltonian in the (orthogonalized) basis


## The gluon chain spectrum

This procedure, if successful, gives us a spectrum of states containing a static quark-antiquark pair, with the spectrum dependent on the quark-antiquark separation $R$. Of course the excited states will be unstable to decay into glueballs. But so are states on Regge trajectories.

First step: only $n=0,1$ constituent gluons, $(M+1) \times(M+1)$ Hamiltonian matrix. The finite basis consists of the zero-gluon state and a set of $M$ one-gluon states.

## Coulomb gauge preliminaries

The classical Hamiltonian:

$$
\begin{aligned}
H & =H_{\text {glue }}+H_{\text {coul }}+H_{\text {matter }} \\
H_{\text {glue }} & =\frac{1}{2} \int d^{3} x\left(\vec{E}^{\mathrm{tr}, a} \cdot \vec{E}^{\mathrm{tr}, a}+\vec{B}^{a} \cdot \vec{B}^{a}\right), \\
H_{\text {coul }} & =\frac{1}{2} \int d^{3} x d^{3} y \rho^{a}(x) K^{a b}(x, y ; A) \rho^{b}(y) \\
K^{a b}(x, y ; A) & =\left[\mathcal{M}^{-1}\left(-\nabla^{2}\right) \mathcal{M}^{-1}\right]_{x y}^{a b}, \\
\mathcal{M} & =-\nabla \cdot \mathcal{D}(A) \\
\rho^{a} & =\rho_{q}^{a}+\rho_{\bar{q}}^{a}+\rho_{g}^{a}, \\
& =g q_{i}^{\dagger}(x) t_{i j}^{a} q_{j}(x)+g \bar{q}_{i}(x) t_{i j}^{a} \bar{q}_{j}^{\dagger}(x)-g f^{a b c} A_{k}^{b}(x) E_{k}^{c}(x)
\end{aligned}
$$

In tree-level formalism we have to expand $K^{a b}(x, y ; A)$ and do a partial resummation into dressed ghost and Coulomb propagators.

## Decomposition of the Coulomb vertex



Diagrammatic notation

|  | static quark |
| :---: | :---: |
| －－－－－－－ | Coulomb |
|  | ghost |
| 心以いい | transverse gluon |
| 10000 | $E-A$ propagator |

These are dressed propagators．

The decomposition involves only RG－invariant combinations of operators．
The approximation is the factorization of the VEV into a product of dressed propagators．

## Decomposition details

The perturbative expansion of the ghost operator $G(A)=1 /(-\nabla \cdot \mathcal{D})$ is the series

$$
G(A)=\frac{1}{\left(-\nabla^{2}\right)} \sum_{n=0}^{\infty} M^{n}, \quad M=g f A_{i} \partial_{i} \frac{1}{\left(-\nabla^{2}\right)}
$$

For the Coulomb operator $K(A)=G(A)\left(-\nabla^{2}\right) G(A)$

$$
\begin{aligned}
K & =\frac{1}{\left(-\nabla^{2}\right)} \sum_{m=0}^{\infty} M^{m}(-\nabla)^{2} \frac{1}{\left(-\nabla^{2}\right)} \sum_{n=0}^{\infty} M^{n} \\
& =\frac{1}{\left(-\nabla^{2}\right)} \sum_{N=0}^{\infty}(N+1) M^{N}
\end{aligned}
$$

Suppose one $M$ operator, call it $M^{*}$, contracts with an $A$ field in the initial or final states.

## Decomposition details ||

Call $K^{A}$ the operator with one $A$ operator contracting with the initial or final state.

$$
K^{A}=\frac{1}{\left(-\nabla^{2}\right)} \sum_{N=0}^{\infty}(N+1) \sum_{m=0}^{N-1} M^{m} M^{*} M^{N-m-1}
$$

With a little bit of combinatorics, one can show that

$$
\begin{align*}
K^{A}= & \frac{1}{\left(-\nabla^{2}\right)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}(m+n+2) M^{m} M^{*} M^{n} \\
= & \frac{1}{\left(-\nabla^{2}\right)} \sum_{m=0}^{\infty}(m+1) M^{m}\left(g f A^{*} \partial\right) \frac{1}{\left(-\nabla^{2}\right)} \sum_{n=0}^{\infty} M^{n} \\
& \quad+\frac{1}{\left(-\nabla^{2}\right)} \sum_{m=0}^{\infty} M^{m}\left(g f A^{*} \partial\right) \frac{1}{\left(-\nabla^{2}\right)} \sum_{n=0}^{\infty}(n+1) M^{n} \\
& =K\left(g f A^{*} \partial\right) G+G\left(g f A^{*} \partial\right) K \tag{3}
\end{align*}
$$

This $K G+G K$ structure generalizes to any number of gluons emerging from the $K(A)$ operator.

## Decomposition of the Coulomb vertex



Diagrammatic notation

|  | static quark |
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## The relevant tree diagrams

$\underline{H_{00}}$
$H_{11}$ kinetic

$H_{11}$ Coulomb (non-planar)
$H_{11}$ KGG term
$H_{11}$ Coulomb (planar)

$\underline{H_{10}}$

$+K G G$ permutations

## truncated basis

The initial/final "blobs" represent one-gluon states in a trucated basis. The construction of one-gluon basis states begins with the one-parameter state

$$
\begin{aligned}
\vec{\Psi} & =\nabla \times\left[\begin{array}{c}
-y \\
x \\
0
\end{array}\right] F(x, y, z)=\left[\begin{array}{c}
-x \partial_{z} F \\
-y \partial_{z} F \\
2 F+x \partial_{x} F+y \partial_{y} F
\end{array}\right] \\
F(x, y, z) & =\exp \left[-\frac{1}{a}\left(\sqrt{x^{2}+y^{2}+z^{2}}+\sqrt{x^{2}+y^{2}+(R-z)^{2}}\right)\right]
\end{aligned}
$$

where $a$ is a variational parameter.

Motivation: $\vec{\Psi}$ is dimensionless, transverse $\vec{\nabla} \cdot \vec{\Psi}=0$, and concentrated in a cylindrical region of radius $a$.


For a large basis, the precise choice of $a$ is irrelevant, but it makes a difference in a small basis.

## basis II

Construct a set of non-orthogonal one-gluon states with the ansatz

$$
\begin{aligned}
\Psi_{k} & =\nabla \times\left[\begin{array}{c}
-y \\
x \\
0
\end{array}\right] F_{n m}(r, z) \\
F_{n m}(r, z) & =f_{n}(z) L_{m}^{1}\left(\frac{4 r}{a}\right) \exp \left[-\frac{1}{a}\left(\sqrt{r^{2}+z^{2}}+\sqrt{r^{2}+(R-z)^{2}}\right)\right]
\end{aligned}
$$

where $L_{m}^{1}$ is an associated Laguerre polynomial, $r^{2}=x^{2}+y^{2}$, and

$$
f_{n}(z)=\left\{\begin{array}{cl}
1 & n=1 \\
\sin \left(\frac{2 \pi n}{R+2 \frac{a}{3}}\left(z+\frac{a}{3}\right)\right) & n>1
\end{array}\right.
$$

## basis III

From the overlaps

$$
O_{j k}=3 C_{F} \int d^{3} x d^{3} y \vec{\Psi}_{j}(x) \cdot \vec{\Psi}_{k}(y) D(x-y)
$$

and Hamiltonian matrix elements

$$
\left\langle\Psi_{j}\right| H\left|\Psi_{k}\right\rangle
$$

we can construct (via stabilized Gram-Schmidt) an orthonormal set of one-gluon states $|\alpha\rangle, \alpha=1,2, \ldots$ (the zero-gluon state is denoted $|0\rangle$ ).

In this truncated basis of zero and one gluon states we can obtain the Hamiltonian matrix elements $\langle\alpha| H|\beta\rangle$.

Then diagonalize $H$ to obtain the spectrum, optimized wrt the parameter $a$.

## The transverse gluon propagator

The Coulomb propagator $\widetilde{K}(R)$ is related to the Coulomb potential via

$$
V_{\text {coul }}(R)=-g^{2} C_{F} \widetilde{K}(R)
$$

Still need the ghost $G(R)$ and equal-times transverse gluon $D_{i j}(R)$ propagators. Because $\vec{\psi}$ is transverse, it suffices to compute

$$
D(R)=\frac{1}{8} \delta_{i j}\left\langle\operatorname{Tr}\left[A_{i}(\mathbf{x}, t) A_{j}(\mathbf{y}, t)\right]\right\rangle
$$

where, on the lattice

$$
A_{i}(\mathbf{x}, t)=\frac{1}{2 i g a}\left(U_{i}(\mathbf{x}, t)-U_{i}^{\dagger}(\mathbf{x}, t)\right)
$$

In lattice simulations, it turns out that the data is very sensitive to lattice volume; more so at higher $\beta$. Our data does seem to have converged for $R \leq 0.7 \mathrm{fm}$ or so on a $30^{4}$ lattice volume, but requires larger volumes at larger $R$.

## volume dependence of the gluon propagator

## Strong lattice volume dependence...


(a) $\beta=5.7$

(c) $\beta=5.9$

(b) $\beta=5.8$

(d) $\beta=6.0$

## gluon propagator II

...but on a $30^{4}$ volume the results converge up to $R \approx 0.7 \mathrm{fm}$.


The data suggests that multiplicative renormalization $Z_{A}$ is almost constant in this range of $\beta$.

## gluon propagator III

We have tried to fit the data to either a massive propagator...

$$
\begin{aligned}
D^{m a s s}(R, m) & =c \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{e^{i \mathbf{k} \cdot \mathbf{x}}}{2\left(k^{2}+m^{2}\right)^{\frac{1}{2}}} \\
& =c \frac{m}{4 \pi^{2} R} K_{1}(m R)
\end{aligned}
$$

...or a Gribov propagator

$$
D^{\text {Grib }}(R, m)=c \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{e^{i k \cdot x}}{2\left(k^{2}+m^{4} / k^{2}\right)^{\frac{1}{2}}}
$$

which can be expressed in terms of MeijerG functions. Neither gives a very good fit.


## gluon propagator IV



In the region where the data has converged, massive and Gribov propagators do not seem to give a good fit to the data. We use instead

$$
D(R)=0.0469 \frac{\exp \left[-5.35 R^{2}+2.35 R\right]}{R^{2}}
$$

## The Ghost Propagator

The lattice ghost propagator

$$
G^{a b}(R)=\left\langle\left(\mathcal{M}^{-1}\right)_{a x, b y}\right\rangle=\delta^{a b} \frac{1}{8}\left\langle\left(\mathcal{M}^{-1}\right)_{c x, c y}\right\rangle
$$

is the inverse of the lattice Faddeev-Popov operator $\mathcal{M}=-\nabla \cdot \mathcal{D}$. A complication is that $\mathcal{M}$ has eight eigenvectors with zero eigenvalues ( $c=1-8$ )

$$
\psi_{a \mathbf{x}}^{(c)}=\frac{1}{L^{3 / 2}} \delta_{a c} \text { with } \mathcal{M}_{a \mathbf{a x}, b \mathbf{y}} \psi_{b \mathbf{y}}^{(c)}=0
$$

associated with a remnant global $\mathrm{SU}(3)$ color symmetry, and is therefore not invertible as it stands.

Solution: invert $\mathcal{M}$ on a subspace orthogonal to these zero modes. This is automatic in momentum space, and can be done numerically.

## ghost II

The ghost propagator in Coulomb gauge has been computed by a German-Japanese collaboration Nakagawa et al. (2009) for the SU(3) group, and by Langfeld and Moyaerts (2004) and Burgio et al. (2012) for SU(2). We will use the SU(3) result found in the infrared region:

$$
G(p)=\left(\frac{d(\beta)}{p^{0.44}}\right) \frac{1}{p^{2}}
$$

which translates, in position space, to

$$
G(R)=\frac{\sqrt{6}}{8} \frac{c(\beta)}{R^{0.56}}
$$

At $\beta=6$, Nakagawa et al. have $c(\beta)=1.63$.

## ghost III

Lattice simulations provide the bare propagators, and in particular the ghost propagator contains a constant factor $c(\beta)$ which is coupling dependent.

In Coulomb gauge, however, the combination $g G A_{\mu}(x)$ is RG invariant, as is $g^{2} K(R)$.
Our spectrum calculation should be RG invariant, at sufficiently weak coupling, providing we use ghost, transverse gluon, and Coulomb propagators computed at the same $\beta$.

There are two questions:
(1) Does the energy of the ground state, computed in our zero + one-gluon basis, rise linearly with $R$ ?
(2) If yes, and we fit $c(\beta)$ to get the right string tension, how close do we get to the value $c(\beta)=1.63$ reported by Nakagawa et al.?

## The Static Quark Potential

Diagonalize $H_{\alpha \beta}$ in the truncated basis, and plot the lowest energy eigenvalue vs. $R$.


A linear potential is obtained, with string tension below the Coulomb string tension.
To get the asymptotic string tension of $(440 \mathrm{MeV})^{2}$, we need $c=2.35$, to be compared to the value $c=1.63$ of Nakagawa et al.. However, this is a variational calculation, and the agreement may improve with an improved ansatz for the basis states.

Spectrum of excited states:


These excitations are far above the glueball threshold, and would rapidly decay to the ground state. It is also likely that two-gluon chain states, in addition to the zero and one-gluon states, would make an important contribution to the excitations.

## eigenstate composition

The minimal energy state at each $R$ is a superposition

$$
\left.\left.|\psi\rangle=c_{0}(R) \mid 0 \text { gluons }\right\rangle_{\bar{q} q}+c_{1}(R) \mid 1 \text { gluon }\right\rangle_{\bar{q} q}
$$

The relative proportions of the zero and one-gluon states are given by $\left|c_{0}^{2}\right|$ and $\left|c_{1}\right|^{2}$ respectively.

The proportions are fairly constant in the range $1.0-2.2 \mathrm{fm}$, with $c_{0}^{2}$ decreasing from 0.77 to 0.72 .

## Conclusions

## The gluon chain model looks promising.

A single constituent gluon, up to $R=2.2 \mathrm{fm}$, seems sufficient to lower the Coulomb string tension by the factor required, without spoiling the linearity of the potential.

At the quantitative level, however:
(1) We need much better lattice Monte Carlo data for the transverse gluon (and, perhaps, ghost) propagators in position space, at larger volumes.
(2) The sensitivity of our results to different ansatze for the truncated basis needs to be explored systematically.

## Next steps

- Gluelump spectra (compare with lattice data)
- Glueballs (compare with lattice pure gauge theory)
- quarkonia (compare with experiment)

The tree diagrams for $H_{m n}$ are essentially the same, although for glueballs and quarkonia it is preferable to work in momentum space.

