A finite-volume local formulation of QCD coupled to QED

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(based on BL, A. Patella, A. Ramos and N. Tantalo, arXiv:1509.01636)

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Outline

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Isospin symmetry and its breaking in QCD

Consider QCD with the three light *degenerate* flavours: $m_u = m_d = m_s = M$

With the quark spinor expressed as

$$\Psi(x) = (\psi_u(x), \psi_d(x), \psi_s(x)) \ ,$$

the Lagrangian

$$\mathcal{L}(x) = \mathcal{L}_{YM}(x) + \overline{\Psi}(x) \left[(i\partial\!\!\!/ + gB\!\!\!/ - M) \otimes \mathbb{I}_{3\times 3} \right] \Psi(x)$$

has a SU(3) isospin global symmetry

However, in nature $m_u < m_d \ll m_s$

- the baryon number $B = \frac{1}{3}\mathbb{I}_{3\times 3}$ is a good quantum number
- in addition, the Cartan subgroup $I_3 = \frac{1}{2}\mathrm{diag}(1,-1,0)$, $Y = \frac{1}{3}\mathrm{diag}(1,1,-2)$ defines global symmetries [$Y = B + S$, S strangeness]
- the SU(2) subgroup containing I_3 is softly broken ($m_u \sim m_d$)

Isospin breaking by QED interactions

If electromagnetic interactions are switched on

$$\mathcal{L}(x) \to \mathcal{L}(x) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - e\overline{\Psi}(x) \not{A}\, Q \Psi(x) \ , \qquad Q = I_3 + \frac{1}{2} Y$$

We observes that

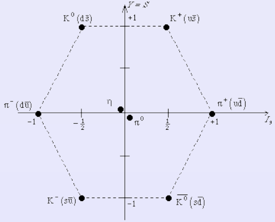
- I_3, Y and B are good quantum numbers also for QED-induced breaking
- Isospin breaking by electromagnetic interactions is perturbative in $\alpha = e^2/(4\pi)$
- In real-world Quantum Chromodynamics, QCD-induced breaking and QED-induced breaking are of the same order of magnitude

Both QCD and QED isospin breaking patterns suggest to classify the spectrum in the ideal situation of unbroken symmetry and to consider breaking effects starting from this ideal case

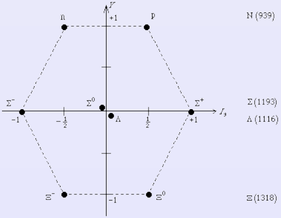
Isospin classification of hadrons

Irreducible representations of isospin flavour (singlet and octet for mesons, singlet, octet and decuplet for baryons) used to classify hadrons

Lowest-lying pseudoscalar meson octet



Lowest-lying spin-1/2 baryon octet



(Figures from http://www.cbooth.staff.shef.ac.uk/phy304/propquark.html)

In the absence of isospin breaking, particles in the same isospin multiplets would be degenerate

Still approximately degeneracy in SU(2) isospin, but breaking effects have important consequences in nature

Why does isospin breaking matters?

For instance, proton-neutron mass splitting

$$\Delta M = m_n - m_p = 1.2933322(4) \text{ MeV}$$

Since $m_n > m_p$

- The hydrogen atom is stable
- The β-decay of the neutron is allowed
- Relative abundance

$$\frac{n_n}{n_p} \simeq e^{-\Delta M/T}$$

important parameter for Big Bang nucleosynthesis

Even small variations of ΔM have dramatic consequences on the evolution of the Universe [see e.g. A. Portelli, arXiv:1505.07057]

Outline

Towards a lattice formulation of QED+QCD

- A first-principle calculation of isospin breaking effects should be possible using lattice gauge theory techniques
- Lattice formulation of QCD well known by now and corroborated by several precision calculations
- Periodic (antiperiodic) boundary conditions for gauge fields (fermions) allow to reduce finite size effects
- Extraction of results on finite lattices in QCD benefits from the existence of a mass gap $\Rightarrow$ finite size corrections are exponentially suppressed
- The absence of a mass gap in QED already indicates that finite size effects are going to be power-law with the lattice size
- However, it is even worse than that: QED on a periodic finite volume is inconsistent with a non-zero charge

We would need to handle this system differently from QCD alone!

Only zero charge on a periodic box

- Classically, it is an immediate consequence of the Gauss law

$$Q = \int_V \rho(x)\,\mathrm{d}V = \int_V \vec{\nabla}\cdot\vec{E}(x)\,\mathrm{d}V = \int_\Sigma \vec{E}(x_{\|})\cdot\mathrm{d}\vec{\sigma} = 0$$

since because of PBC for each $\vec{E}\cdot\mathrm{d}\vec{\sigma}$ contribution to the integral there is another with opposite sign

- in QED the generator of gauge transformations in the Schrödinger picture is

$$\hat{G}(x) = \vec{\nabla}\cdot\hat{\vec{E}}(x) - \hat{\rho}(x)$$

Charge operator

$$\hat{Q} = \int_V \left(\vec{\nabla}\cdot\hat{\vec{E}}(x) - \hat{G}(x)\right)\,\mathrm{d}V = -\int_V \hat{G}(x)\,\mathrm{d}V$$

if $|\psi\rangle$ is Physical, $\hat{G}(x)|\psi\rangle = 0 \Rightarrow Q|\psi\rangle = 0$

Absence of charge connected to gauge invariance

Gauge fixing and zero modes

Define the gauge-fixed field $\psi_{GF}(x) \Rightarrow$ Action still invariant for $\psi_{GF}(x) \rightarrow e^{i\alpha}\psi_{GF}(x)$

As a consequence, the action is left invariant by

$$\begin{aligned} \psi(x) &\rightarrow e^{\sum_\rho \frac{2\pi n_\rho}{L_\rho} x_\rho} \psi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) + \frac{2\pi n_\rho}{L_\rho} \end{aligned}$$

which do survive gauge fixing

Hence, because of these *zero modes*, correlators of the form $\langle \overline{\psi}_{GF}(x)\psi_{GF}(y)\rangle = 0$ if $x \neq y$

At finite volume

Gauss law $\Leftrightarrow$ gauge invariance $\Leftrightarrow$ zero models $\Leftrightarrow$ large gauge transformations

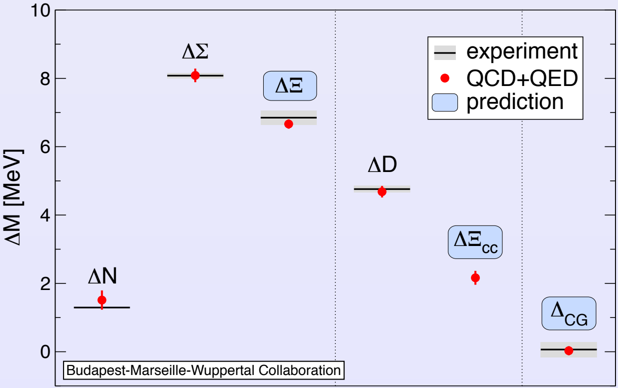
QED+QCD simulations: the story so far

- **Electroquenching** [Various authors]
  The back reaction of quarks on photons is neglected
  $\hookrightarrow$ Uncontrolled systematic error (but expansion in α possible)

- **Non-relativistic QED** [e.g. Lee and Tiburzi, arXiv:1508.04165]
  $\hookrightarrow$ Antiparticle contribution needs to be kept into account

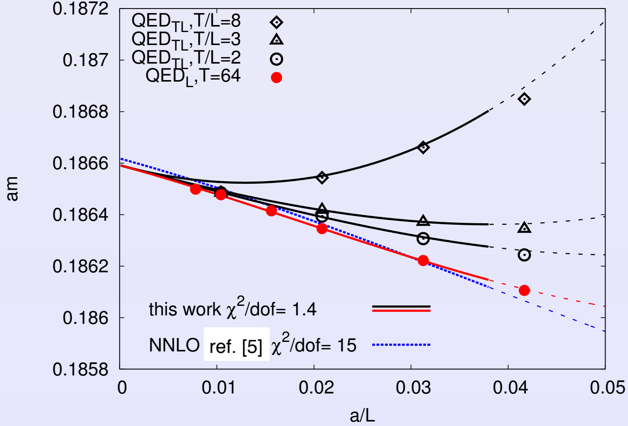
- **QED$_L$** [Hayakawa and Uno, Prog. Theor. Phys. 120(3), 413 (2008)]
  Zero modes are removed time slice by time slice
  So far gives the best numerical results [Borsanyi et al., Science 347 (2015)]
  $\hookrightarrow$ Very large lattices are required and the non-locality of the theory generates difficult to quantify systematic effects

The BMWc calculation (QED$_L$)



[Borsanyi et al., Science 347 (2015)]

Infinite volume extrapolation (QED_L)



[Borsanyi et al., Science 347 (2015)]

QED+QCD simulations: new proposals

Different approaches needed for robust results

Recent promising proposals all at an early stage

- **Twist averaging**[Lehner, Izubuki, arXiv:1503.04395]
  QCD fields replicated, twisted fermions and QED in infinite volume
  $\hookrightarrow$ Requires efficient sampling over momenta of the photon

- **Massive QED** [Endres, Schindler, Tiburzi, Walker-Loud, arXiv:1507.0891]
  A small mass is given to the photon and the zero-mass limit is taken
  $\hookrightarrow$ Promising electroquenched results, but the extrapolation to zero mass might be hard

- **QCD+QED on $C^\star$ lattices** [Lucini, Patella, Ramos, Tantalo, in preparation]
  Local formulation at finite volume exploiting charge conjugation symmetry for the BC [Polley, 1991]

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Outline

Locality

A Lorentz-invariant QFT with a Lagrangian $\mathcal{L}(x)$ depending on the fields and their first derivatives, with fields evolving according to the equations derived from the minimal action principle, respects *microcausality*: if A and B are polynomial in the fields and their derivatives

$$[A(\vec{x},t), B(\vec{y},t)] = 0 \qquad \vec{x} \neq \vec{y}$$

Consequences are

- Existence of antiparticles
- Renormalisation by power counting
- Volume-independence of renormalisation constants
- Operator Product Expansion
- No UV- IR mixing
- ...

Those properties underpin Lattice Gauge Theory methods (e.g., combined extrapolation in the volume and in the lattice spacing possible because of UV-IR separation)

In non-local theories, the validity of those properties needs to be proved

QED$_C$

QED$_C$ is QED in a finite box closed with $C^\star$ boundary conditions

With C charge conjugation matrix, $C^\star$ boundary conditions defined by [Polley, 1991]

$$A_\mu(x + L_k\hat{k}) \to -A_\mu(x) \ , \qquad \psi(x + L_k\hat{k}) \to C^{-1}\overline{\psi}^T(x) \ , \qquad \overline{\psi}(x + L_k\hat{k}) \to -\psi^T(x)C$$

- Based on an invariance of the action ⇒ respects the spacetime symmetries of the system (charge discussed later)
- Gauge fields free from zero modes
- Classically there is no obstruction to the Gauss law
- Hamiltonian proof of existence of single-charge states given by Polley

Charge and mass in QED$_C$

Multiple charges in QED$_C$ require a dedicated discussion

In QED at infinite volume the charge is defined through a global U(1) invariance and the electric charge takes values in $\mathbb{Z}$

In QED$_C$ possible gauge transformations are

$$\Omega(x) = \pm e^{i\alpha(x)} \ , \qquad \alpha(x + L_k \hat{k}) = -\alpha(x)$$

Usual U(1) gauge group, global symmetry group $\mathbb{Z}_2 \Rightarrow Q$ no longer is a good quantum number, but $(-1)^Q$ is conserved
$\hookrightarrow$ States with even charge mix with states with even charge and states with odd charge mix with states with odd charge

It is still possible to define the mass of the electron under the *physically motivated assumption* that the electron itself is the lightest state in the sector with negative charge

Charged operators and mass

Gauge-fixed definition of ψ possible, but gauge-covariant definition desirable

Solution known: provide the fermion field with a Mandelstam string, e.g.

$$\Psi(\vec{x},t) = e^{-i\int \mathrm{d}^3 y\ \Phi(\vec{x}-\vec{y})\vec{\nabla}_y\cdot\vec{A}(\vec{y},t)}\psi(\vec{x},t)$$

with $\Phi(\vec{x},t)$ electric potential generated by a unit charge at $\vec{x}$ in a $C^\star$ box:

$$\begin{aligned} \nabla^2\Phi(\vec{x}) &= \delta^3(\vec{x}) \\ \Phi(\vec{x}+L_k\hat{k}) &= -\Phi(\vec{x}) \end{aligned}$$

In the Coulomb gauge, $\Psi(\vec{x},t) = \psi(\vec{x},t) \Rightarrow$ The construction is a covariant generalisation of the Coulomb gauge

Construction not unique, argument more general: any Ψ such that

- Ψ is invariant under local gauge transformations
- Ψ picks up the defining U(1) factor under a global gauge transformation

can be used

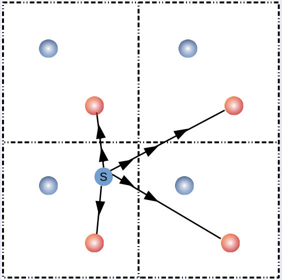
We can define the electron mass through

$$\left\langle \overline{\Psi}(\vec{0},0)\left(\sum_{\vec{x}}\Psi(\vec{x},t)\right)\right\rangle \overset{t\to\infty}{\to} ce^{-m_{el}t}$$

Outline

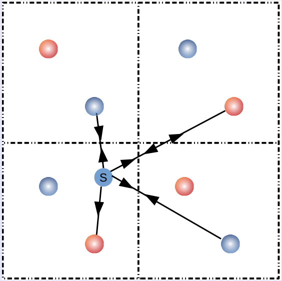
A classical view on charge sectors

Field transmitting walls with PBC



Charges are replicated in sublattices

Field inverting walls with $C^\star$ BC



Charges are reflected in sublattices

Classically $C^\star$ boundaries act as a charge-conjugation mirror of the charge
$\hookrightarrow$ Each unit charge is seen as its anticharge through the boundary

Through the boundary, arrows on the field are flipped by $C^\star$ BC and field lines can connect charges with the same sign

Recovering charge superselection

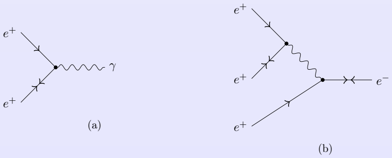
On a $C^\star$ box (fixed gauge, $\langle\vec{n}\rangle = \sum_k n_k \mod 2$, n_k wrapping in direction k)

$$\langle\psi(x)\bar{\psi}(y)\rangle = \sum_{\langle\vec{n}\rangle=0} S(x-y+n_k L_k \hat{k})$$

$$\langle\psi(x)\psi^T(y)\rangle = -\sum_{\langle\vec{n}\rangle=1} S(x-y+n_k L_k \hat{k})C^{-1}$$

$$\langle\bar{\psi}^T(x)\bar{\psi}(y)\rangle = \sum_{\langle\vec{n}\rangle=1} CS(x-y+n_k L_k \hat{k})$$

Charge-violating Feynman diagrams



Effects exponentially suppressed with the lattice size

$$\langle\psi(x)\psi^T(y)\rangle \sim \langle\bar{\psi}^T(x)\bar{\psi}(y)\rangle \sim \left(\frac{m}{L}\right)^{\frac{3}{2}} e^{-mL}$$

Flavour symmetry in (QCD + QED)$_C$

QED$_C$ can be coupled to QCD in the usual way, with the *proviso* that since matter obeys $C^\star$ BC, also the SU(3) gauge fields need to satisfy $C^\star$ BC $\Rightarrow$ (QCD + QED)$_C$

- Internal quantum numbers:

$$F_f \ , \qquad F = \sum_{f=1}^{N_f} F_f \ , \qquad Q = \sum_{f=1}^{N_f} q_f F_f \ , \qquad B = F/3 \ , \qquad \Delta F_f = 0 \bmod 2$$

- For a large enough box, confinement implies

$$\Delta Q = 0 \bmod 2 \ , \qquad \Delta B = 0 \bmod 2 \ , \qquad \Delta F = 0 \bmod 6$$

$\hookrightarrow$ In the presence of confinement

- Flavour violation in $C^\star$ BC is such that fractional charges never emerge
- In pure QCD, because of the mass gap, again we expect processes that violate these quantum numbers to be *exponentially suppressed*
- However, we do have photons in this theory: are they a problem?

Mixing of hadrons in a $C^\star$ box

The importance of mixing effects can be evaluated in an effective theory of hadrons on a $C^\star$ box

- All possible QCD processes included in vertices
- The vertices in a $C^\star$ box are the same as in infinite volume
- Propagators with an even wrapping do not violate flavour
- Propagators with an odd wrapping violate flavour
- Allowed mixing are related to (virtually) allowed physical processes
  Example: mixing $\Xi^- - p$ possible (for instance) by virtue of

$$\Xi^- \to \left(\Lambda^0 + K^-\right)^* , \qquad \left(\Lambda^0 + K^+\right)^* \to p$$

Results

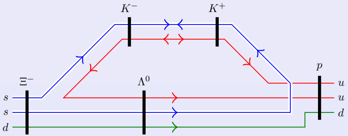
- Finite-size effects giving rise to spurious hadron mixing suppressed exponentially
- The corresponding processes are typically $\mathcal{O}(10^{-8} - 10^{-10})$ in physically interesting cases

Even with mixing, masses of lowest-lying particles (e.g. proton, neutron) can be extracted by looking at the large-time behaviour of effective masses at finite (large) volume

Flavour-violating diagrams

Example: Ξ^- mixing possibilities

Simple case: mixing with p



Amplitude of order $e^{-m_K L}$, mixing suppression factor $e^{-2m_K L}$

Other possibilities: mixing with $p+\gamma+\gamma$, $n+\pi^+(+\gamma+\dots)$

One can prove that mixing suppression never larger than $e^{-2\mu}$,

$$\mu = \left[M_{K^\pm}^2 - \left(\frac{M_{\Xi^-}^2 - M_{\Lambda^0}^2 + M_{K^\pm}^2}{2M_{\Xi^-}} \right)^2 \right]^{1/2} ,$$

($\mathcal{O}(10^{-10})$ in current settings for lattice calculations)

Outline

Finite size corrections in $(\text{QCD+QED})_C$

At first order in α

$$\frac{m(L)-m}{m} = \underbrace{\left.\frac{\Delta m(L)}{m}\right|_{universal} + \left.\frac{\Delta m(L)}{m}\right|_{structure-dependent}}_{\text{QED only}} + \left.\frac{\Delta m(L)}{m}\right|_{QCD+QED}$$

From first principles we can prove that

$$\left.\frac{\Delta m(L)}{m}\right|_{universal} = \frac{e^2}{4\pi}\left\{\frac{q^2\xi(1)}{2mL} + \frac{q^2\xi(2)}{\pi(mL)^2}\right\}$$

$$\left.\frac{\Delta m(L)}{m}\right|_{structure-dependent} = -\frac{e^2}{16\pi^2 m}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell(2\ell)!}{\ell! L^{2+2\ell}}\mathcal{T}_\ell\xi(2+2\ell)$$

$$\left.\frac{\Delta m(L)}{m}\right|_{QCD+QED} = \mathcal{O}(e^{-m_\pi L}) + \alpha\mathcal{O}(e^{-\frac{\sqrt{3}}{2}m_\pi L})$$

Finite size corrections in (QCD+QED)$_C$

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$$\left.\frac{\Delta m(L)}{m}\right|_{QCD+QED} = \mathcal{O}(e^{-m_\pi L}) + \alpha\mathcal{O}(e^{-\frac{\sqrt{3}}{2}m_\pi L})$$ negligible

Let's analyse the first two in more detail

The universal corrections

$$\left.\frac{\Delta m(L)}{m}\right|_{universal} = \frac{e^2}{4\pi}\left\{\frac{q^2\xi(1)}{2mL} + \frac{q^2\xi(2)}{\pi(mL)^2}\right\}$$

- Fixed by Ward identities $\Rightarrow$ Independent of the spin and of the structure of the hadron
- Effects of the $C^\star$ BC encoded in the generalised zeta function

$$\xi(s) = \sum_{\vec{n}\neq\vec{0}} \frac{(-1)^{\sum_{j\in C^\star} n_j}}{|\vec{n}|^s} \ .$$

	$1C^\star$	$2C^\star$	$3C^\star$
$\xi(1)$	-0.77438614142	-1.4803898065	-1.7475645946
$\xi(2)$	-0.30138022444	-1.8300453641	-2.5193561521
$\xi(4)$	0.68922257439	-2.1568872986	-3.8631638072

The structure-dependent corrections

$$\left.\frac{\Delta m(L)}{m}\right|_{structure-dependent} = -\frac{e^2}{16\pi^2 m}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell(2\ell)!}{\ell! L^{2+2\ell}}\mathcal{T}_\ell\,\xi(2+2\ell)$$

- Starts at $\mathcal{O}(1/L^4)$ (no term in the whole correction is of $\mathcal{O}(1/L^3)$!)
- Runs only in even powers of $1/L$
- Boundary conditions still enter through ξ
- Details of the structure and spin encoded in $\mathcal{T}_\ell$
- With $T_{\mu\mu}(ik,\vec{k})$ forward Compton scattering amplitude of a photon of momentum $\vec{k}$ on the hadron at rest

$$\mathcal{T}_\ell = \left.\frac{\mathrm{d}^\ell}{\mathrm{d}(k^2)^\ell}T_{\mu\mu}(ik,\vec{k})\right|_{k^2=0}$$

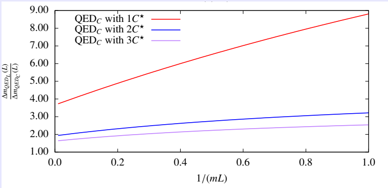
Comparison with QED$_L$

$$\left.\frac{\Delta m(L)}{m}\right|_{QED_C} = \frac{e^2}{4\pi}\left\{\frac{q^2\xi(1)}{2mL} + \frac{q^2\xi(2)}{\pi(mL)^2} - \frac{1}{4\pi^2 m}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell(2\ell)!}{\ell! L^{2+2\ell}}\mathcal{T}_\ell\xi(2+2\ell)\right\} + \dots$$

vs.

$$\left.\frac{\Delta m(L)}{m}\right|_{QED_L} = \frac{e^2}{4\pi}\left\{-\frac{q^2 k}{2mL} - \frac{q^2 k}{(mL)^2} + \mathcal{O}\left(\frac{1}{L^3}\right)\right\} + \dots$$

- In QED$_L$, structure-dependent corrections start at $\mathcal{O}(1/L^3)$, while in QED$_C$ they start at $\mathcal{O}(1/L^4)$ (and run in even powers of $1/L$)
- structure-independent corrections smaller in QED$_C$



Outline

U(1) gauge theory on the lattice

1. Non-compact formulation

$$S_\gamma = \frac{1}{2e^2} \sum_{x,\mu<\nu} (F_{\mu\nu}(x))^2 \ ,$$

$$F_{\mu\nu}(x) = A_\mu(x) + A_\nu(x+\hat{\mu}) - A_\mu(x+\hat{\nu}) - A_\nu(x)$$

   - Gauge-invariant
   - Requires to fix the gauge in simulations $\Rightarrow$ Not a problem

2. Compact formulation

$$S_\gamma = \frac{1}{e^2} \sum_{x,\mu<\nu} (1 - \cos\Theta_{\mu\nu}(x)) \ ,$$

$$\Theta_{\mu\nu}(x) = \theta_\mu(x) + \theta_\nu(x+\hat{\mu}) - \theta_\mu(x+\hat{\nu}) - \theta_\nu(x) \ , \qquad \theta_\mu(x) = \arg\left(e^{iA_\mu(x)}\right)$$

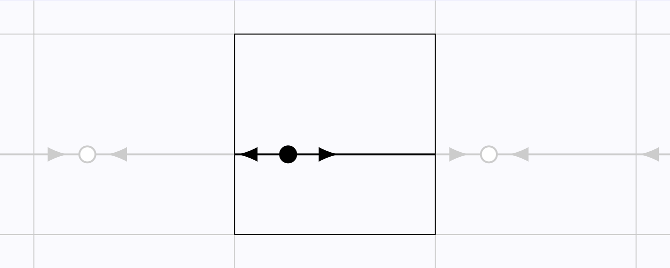
   - Gauge-invariant
   - Does not require to fix a gauge
   - More natural on the lattice

Focus on compact formulation

Lattice formulation of charged operators

- A Mandelstam string localised on a line gives the easiest case for the discretisation of an electrically charged operator
- In $C^\star$ BC a possibility is

$$\Psi_{\mathbf{s}}(x) = e^{-\frac{\imath q}{2}\int_{-x_k}^{0} ds\ A_k(x+s\hat{k})}\psi(x)e^{\frac{\imath q}{2}\int_{0}^{L-x_k} ds\ A_k(x+s\hat{k})} .$$



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$$\Psi_{\mathbf{s}}(x) = e^{-\frac{\imath q}{2} \int_{-x_k}^{0} ds\, A_k(x+s\hat{k})} \psi(x) e^{\frac{\imath q}{2} \int_{0}^{L-x_k} ds\, A_k(x+s\hat{k})} .$$

- Solve the problem of the square root of the link by considering a theory in which the physical charge comes in multiples of two

$$S_\gamma = \frac{4}{e^2} \sum_{x,\mu<\nu} \left(1 - \cos \Theta_{\mu\nu}(x)\right)$$

- Charged operator on a lattice given by

$$\Psi_{\mathbf{s}}(x) = \prod_{s=-x_k}^{-1} U(x+s\hat{k},k)^{-1}\, \psi(x) \prod_{s=0}^{L-x_k-1} U(x+s\hat{k},k)\ , \qquad U(i,\mu) = e^{i\theta_\mu(i)}$$

The lattice action of $(\text{QCD+QED})_C$

Full Action $S = S_g + S_\gamma + S_m$

Wilson action for QCD

$$S_g = \frac{2}{g^2} \sum_{x,\mu<\nu} \text{Tr}\left(1 - Q_{\mu\nu}(x)\right)$$

$$Q_{\mu\nu}(x) = \mathcal{R}\text{e}\left(V(x,\mu)V(x+\hat{\mu},\nu)V(x+\hat{\nu},\mu)^{-1}V(x,\nu)^{-1}\right) \,, \qquad V(x,\mu) \in \text{SU}(3)$$

Fractionary charges of quarks change the normalisation in S_γ

$$S_\gamma = \frac{36}{e^2} \sum_{x,\mu<\nu} \left(1 - \cos\Theta_{\mu\nu}(x)\right)$$

Matter-field interaction accounted for by the Dirac operator

$$S_m = \sum_f \sum_x \bar{\psi}_f(x) D_f[U,V] \psi_f(x)$$

Standard matter-field coupling for the QCD interaction, parallel transport U^2 rather than U in the QED part

No new sign problem in (QCD+QED)$_C$

- In QCD with non-degenerate quark mass, away from the continuum limit, the Dirac operator with Wilson fermions can have negative eigenvalues
- However, these are lattice artefacts $\Rightarrow$ mild sign problem that can be dealt with
- Fermionic contribution to the path integral with $C^\star$ BC

$$\int_{C^\star \text{ BC}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\bar{\psi}D[V]\psi} = \text{Pf}_K \, C D_{\mathcal{J}}$$

with the Pfaffian defined by

$$(\text{Pf}_K \, C D_{\mathcal{J}})^2 = \text{Det}_K \, C D_{\mathcal{J}} = \text{Det}_K \, D_{\mathcal{J}} \; ,$$

and the indices K and J hiding modifications to the Dirac operators induced by $C^\star$ BC (represented by linear operators in the SU(2) algebra acting on Majorana components of the spinors)
- One can prove that the reality of $\text{Pf}_K \, C D_{\mathcal{J}}$ is a consequence of the reality of $D_{\mathcal{J}}$
- At the same time, $\text{Pf}_K \, C D_{\mathcal{J}} < 0 \Leftrightarrow D_{\mathcal{J}} < 0$ No new sign problem!

Outline

Conclusions

- $C^\star$ BC enable us to provide a local formulation of QCD+QED on a finite volume
- The price to pay is a partial violation of flavour and charge conservation, but
  - charged states relevant for isospin breaking effects can still be defined
  - conservation laws are recovered exponentially fast in some mass parameter related to the m_K mass
  - in current lattice setups, the effect of the violations are negligible
- Finite-volume QED corrections to hadron masses are **significantly smaller** than in QED$_L$ and can be understood in terms of physical considerations $\Rightarrow$ no spurious effect
- The framework is amenable to lattice simulations $\Rightarrow$ non-perturbative results can be extracted
- In passing by: developed novel analytical techniques for determining finite volume effects $\Rightarrow$ potentially useful for other applications

Next step: Monte Carlo simulations to see how this framework performs in practice and get robust determinations of isospin breaking effects