# Lattice Studies of QCD Green's Functions beyond the Gribov Horizon

Tereza Mendes

in collaboration with Attilio Cucchieri

Instituto de Física de São Carlos University of São Paulo

We use lattice gauge theory to study (in Landau gauge) the consequences of the restriction of the path integral to the Gribov region  $\Omega$ , as prescribed in the so-called Gribov-Zwanziger confinement scenario.

We use lattice gauge theory to study (in Landau gauge) the consequences of the restriction of the path integral to the Gribov region  $\Omega$ , as prescribed in the so-called Gribov-Zwanziger confinement scenario.

Lattice approach allows access to the (physically relevant) gauge configurations, which are used to compute "observables" (e.g. propagators and vertices)  $\Rightarrow$  try to extract the most information from them!

We use lattice gauge theory to study (in Landau gauge) the consequences of the restriction of the path integral to the Gribov region  $\Omega$ , as prescribed in the so-called Gribov-Zwanziger confinement scenario.

Lattice approach allows access to the (physically relevant) gauge configurations, which are used to compute "observables" (e.g. propagators and vertices)  $\Rightarrow$  try to extract the most information from them!

In fact, we can also visit nearby configurations, that likely do not have significant statistical weight, and see what information they carry

We use lattice gauge theory to study (in Landau gauge) the consequences of the restriction of the path integral to the Gribov region  $\Omega$ , as prescribed in the so-called Gribov-Zwanziger confinement scenario.

Lattice approach allows access to the (physically relevant) gauge configurations, which are used to compute "observables" (e.g. propagators and vertices)  $\Rightarrow$  try to extract the most information from them!

In fact, we can also visit nearby configurations, that likely do not have significant statistical weight, and see what information they carry

By conveniently crossing the Gribov horizon, we determine geometric properties of  $\Omega$ , and investigate why the predicted ghost enhancement is not found in lattice simulations

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \det K(U)}{Z}$$

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \det K(U)}{Z}$$

Very complicated (high-dimensional) integral to compute!

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \det K(U)}{Z}$$

Very complicated (high-dimensional) integral to compute!

 $\Rightarrow$  Monte Carlo simulations: sample representative gauge configurations, then compute O and take average

Gauge action written in terms of oriented plaquettes formed by the link variables  $U_{x,\mu}$ , which are group elements

- Gauge action written in terms of oriented plaquettes formed by the link variables  $U_{x,\mu}$ , which are group elements
- under gauge transformations:  $U_{x,\mu} \rightarrow g(x) U_{x,\mu} g^{\dagger}(x + \mu)$ , where  $g \in SU(3) \Rightarrow$  closed loops are gauge-invariant quantities

- Gauge action written in terms of oriented plaquettes formed by the link variables  $U_{x,\mu}$ , which are group elements
- under gauge transformations:  $U_{x,\mu} \rightarrow g(x) U_{x,\mu} g^{\dagger}(x + \mu)$ , where  $g \in SU(3) \Rightarrow$  closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing

- Gauge action written in terms of oriented plaquettes formed by the link variables  $U_{x,\mu}$ , which are group elements
- under gauge transformations:  $U_{x,\mu} \rightarrow g(x) U_{x,\mu} g^{\dagger}(x + \mu)$ , where  $g \in SU(3) \Rightarrow$  closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing
- when gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix

- Gauge action written in terms of oriented plaquettes formed by the link variables  $U_{x,\mu}$ , which are group elements
- under gauge transformations:  $U_{x,\mu} \rightarrow g(x) U_{x,\mu} g^{\dagger}(x + \mu)$ , where  $g \in SU(3) \Rightarrow$  closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing
- when gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix
- get FP matrix without considering ghost fields explicitly

- Gauge action written in terms of oriented plaquettes formed by the link variables  $U_{x,\mu}$ , which are group elements
- under gauge transformations:  $U_{x,\mu} \to g(x) U_{x,\mu} g^{\dagger}(x + \mu)$ , where  $g \in SU(3) \Rightarrow$  closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing
- when gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix
- get FP matrix without considering ghost fields explicitly
- Lattice momenta given by  $\hat{p}_{\mu} = 2 \sin (\pi n_{\mu}/N)$  with  $n_{\mu} = 0, 1, \dots, N/2 \iff p_{min} \sim 2\pi/(a N) = 2\pi/L,$  $p_{max} = 4/a$  in physical units

Formulated for Landau gauge, predicts gluon propagator  $D(p^2)$  suppressed in the IR limit, a result that may be viewed as an indication of gluon confinement

Formulated for Landau gauge, predicts gluon propagator  $D(p^2)$  suppressed in the IR limit, a result that may be viewed as an indication of gluon confinement

Long range effects are felt in the ghost propagator G(p):

Formulated for Landau gauge, predicts gluon propagator  $D(p^2)$  suppressed in the IR limit, a result that may be viewed as an indication of gluon confinement

Long range effects are felt in the ghost propagator G(p):

Infinite volume favors configurations on the first Gribov horizon, where minimum nonzero eigenvalue λ<sub>min</sub> of Faddeev-Popov operator M goes to zero

Formulated for Landau gauge, predicts gluon propagator  $D(p^2)$  suppressed in the IR limit, a result that may be viewed as an indication of gluon confinement

Long range effects are felt in the ghost propagator G(p):

- Infinite volume favors configurations on the first Gribov horizon, where minimum nonzero eigenvalue λ<sub>min</sub> of Faddeev-Popov operator M goes to zero
- In turn, G(p) should be IR enhanced, introducing long-range effects, which are related to the color-confinement mechanism

#### **Ghost Enhancement (I)**

Ghost-enhanced scenario natural in Coulomb gauge. Since  $(\partial_i A_i)^a = 0$ , the color-electric field is decomposed as  $E_i^{tr} - \partial_i \phi(\vec{x}, t)$  and the classical (non-Abelian) Gauss law

$$(D_i E_i)^a(\vec{x}, t) = \rho^a_{quark}(\vec{x}, t)$$

is written for a color-Coulomb potential in terms of Faddeev-Popov operator:  $\mathcal{M}\phi^a(\vec{x},t) = \rho^a(\vec{x},t)$ , where  $G^{-1} \sim \mathcal{M} = -D_i\partial_i$ . In momentum space

$$\phi^{a}(\vec{x},t) \approx \int d^{3}p \int d^{3}y \ G(\vec{p},t) \exp[i\vec{p}\cdot(\vec{x}-\vec{y})] \ \rho^{a}(\vec{y},t)$$

IR divergence of ghost propagator  $G(\vec{p}, t)$  as  $1/p^4$  leads to linearly rising potential

QCD–TNT 4

## **Ghost Enhancement (II)**

Gribov's restriction beyond quantization using Faddeev-Popov (FP) method implies taking a minimal gauge, defined by a minimizing functional in terms of gauge fields and gauge transformation

 $\Rightarrow$  FP operator (second variation of functional) has non-negative eigenvalues. First Gribov horizon  $\partial \Omega$  approached in infinite-volume limit, implying ghost enhancement



#### **Ghost Propagator on the Lattice**

Ghost fields are introduced as one evaluates functional integrals by the Faddeev-Popov method, which restricts the space of configurations through a gauge-fixing condition. The ghosts are unphysical particles, since they correspond to anti-commuting fields with spin zero.

On the lattice, the (minimal) Landau gauge is imposed as a minimization problem and the ghost propagator is given by

$$G(p) = \frac{1}{N_c^2 - 1} \sum_{x, y, a} \frac{e^{-2\pi i \, k \cdot (x - y)}}{V} \left\langle \mathcal{M}^{-1}(a, x; a, y) \right\rangle,$$

where the Faddeev-Popov (FP) matrix  $\mathcal{M}$  is obtained from the second variation of the minimizing functional.

Early simulations: Suman & Schilling, PLB 1996; Cucchieri, NPB 1997

QCD–TNT 4

#### **Lattice Landau Gauge**

The lattice Landau gauge is imposed by minimizing the functional

$$S[U;\omega] = -\sum_{x,\mu} Tr \ U^{\omega}_{\mu}(x) ,$$

where  $\omega(x) \in SU(N)$  and  $U^{\omega}_{\mu}(x) = \omega(x) U_{\mu}(x) \omega^{\dagger}(x + a e_{\mu})$  is the lattice gauge transformation.

By considering the relations  $U_{\mu}(x) = e^{i a g_0 A_{\mu}(x)}$  and  $\omega(x) = e^{i \tau \theta(x)}$ , we can expand  $S[U; \omega]$  (for small  $\tau$ ):

$$S[U;\omega] = S[U;\mathbb{1}] + \tau S'[U;\mathbb{1}](b,x) \theta^{b}(x)$$

+ 
$$\frac{\tau^2}{2} \theta^b(x) S''[U; \mathbb{L}](b, x; c, y) \theta^c(y) + \dots$$

where  $S''[U; \mathbb{1}](b, x; c, y) = \mathcal{M}(b, x; c, y)[A]$  is a lattice discretization of the Faddeev-Popov operator  $-D \cdot \partial$ .

Note: large-lattice results ( $L \approx 27$  fm)

Gluon sector:

Gluon propagator is suppressed in the limit  $p \rightarrow 0$ , while the real-space propagator violates reflection positivity

Note: large-lattice results ( $L \approx 27$  fm)

Gluon sector:

- Gluon propagator is suppressed in the limit  $p \rightarrow 0$ , while the real-space propagator violates reflection positivity
- D(0) > 0 (good fit to e.g. Gribov-Stingl form)

Note: large-lattice results ( $L \approx 27$  fm)

Gluon sector:

Gluon propagator is suppressed in the limit  $p \rightarrow 0$ , while the real-space propagator violates reflection positivity

**D**(0) > 0 (good fit to e.g. Gribov-Stingl form)

Ghost sector:

 $\lambda_{min} \rightarrow 0$  with the volume

Note: large-lattice results ( $L \approx 27$  fm)

Gluon sector:

Gluon propagator is suppressed in the limit  $p \rightarrow 0$ , while the real-space propagator violates reflection positivity

**D**(0) > 0 (good fit to e.g. Gribov-Stingl form)

Ghost sector:

- $\lambda_{min} \rightarrow 0$  with the volume
- $\square$  G(p) shows no enhancement in the IR

Note: large-lattice results ( $L \approx 27$  fm)

Gluon sector:

Gluon propagator is suppressed in the limit  $p \rightarrow 0$ , while the real-space propagator violates reflection positivity

**D**(0) > 0 (good fit to e.g. Gribov-Stingl form)

Ghost sector:

 $\lambda_{min} \rightarrow 0$  with the volume

 $\square$  G(p) shows no enhancement in the IR

Consistent with so-called massive solution of DSEs and refined GZ scenario. Not consistent with scaling solution

QCD-TNT 4

#### **Ghost Propagator Results**

Fit of the ghost dressing function  $p^2G(p^2)$  as a function of  $p^2$  (in GeV) for the 4d case ( $\beta = 2.2$  with volume  $80^4$ ). We find that  $p^2G(p^2)$  is best fitted by the form  $p^2G(p^2) = a - b[\log(1 + cp^2) + dp^2]/(1 + p^2)$ , with



$$a = 4.32(2),$$
  
 $b = 0.38(1) \, GeV^2,$   
 $c = 80(10) \, GeV^{-2},$   
 $d = 8.2(3) \, GeV^{-2}.$ 

In IR limit  $p^2G(p^2) \sim a$ .



## **Upper and Lower Bounds for** G(p)

On the lattice, the **ghost propagator** is given by

$$G(p) = \frac{1}{N_c^2 - 1} \sum_{x, y, a} \frac{e^{-2\pi i \, k \cdot (x - y)}}{V} \mathcal{M}^{-1}(a, x; a, y)$$

$$= \frac{1}{N_c^2 - 1} \sum_{i, \lambda_i \neq 0} \frac{1}{\lambda_i} \sum_a |\widetilde{\psi}_i(a, p)|^2 ,$$

where  $\psi_i(a, x)$  and  $\lambda_i$  are the eigenvectors and eigenvalues of the FP matrix. Then, one can prove (A.Cucchieri, TM, PRD 78, 2008) that

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_1} \sum_{a} |\widetilde{\psi}_1(a, p)|^2 \leq G(p) \leq \frac{1}{\lambda_1}.$$

If  $\lambda_1 \equiv \lambda_{min}$  behaves as  $L^{-\alpha}$  in the infinite-volume limit,  $\alpha > 2$  is a necessary condition to obtain an IR-enhanced ghost propagator G(p).

## **Upper bound for** $G(p_{min})$



## **The Infinite-Volume Limit**

One can check if lattice data support  $\lambda_{min}[A] \to 0$  in the infinite-volume limit  $\Longrightarrow A \in \partial \Omega$  (Gribov horizon).



Infinite-volume limit extrapolation  $\lambda_{min}[A] \sim L^c$  for the 3*d* SU(2) case (A.Cucchieri, A.Maas, TM, PRD 74, 2006). Similarly in 4*d*.

#### **The Infinite-Volume Limit (II)**

We thus see that, as the infinite-volume limit is approached, the sampled configurations (inside  $\Omega$  = region for which  $\mathcal{M}$  is positive semi-definite) are closer and closer to the first Gribov horizon  $\partial \Omega$ 



## **The Infinite-Volume Limit (II)**

We thus see that, as the infinite-volume limit is approached, the sampled configurations (inside  $\Omega$  = region for which  $\mathcal{M}$  is positive semi-definite) are closer and closer to the first Gribov horizon  $\partial \Omega$ 



Can we learn more about the geometry of this region?

# **Reaching (and Crossing!) the Horizon**



How many roads have I wondered? None, and each my own Behind me the bridges have crumbled No question of return

Nowhere to go but the horizon where, then, will I call my home?

The Same Song, Susheela Raman

# **Reaching (and Crossing!) the Horizon**



How many roads have I wondered? None, and each my own Behind me the bridges have crumbled No question of return

Nowhere to go but the horizon where, then, will I call my home?

The Same Song, Susheela Raman

— They say that communism is just over the horizon. What's a horizon?

— A horizon is an imaginary line which continues to recede as you approach it.

Russian joke from Khrushchev's time

#### **Relating** $\lambda_{min}$ **and Geometry**

It is generally accepted that

A. Cucchieri, T.M., PRD 2013

At very large volumes the functional integration gets concentrated on the boundary  $\partial \Omega$  of the first Gribov region  $\Omega$ .

But

The key point seems to be the rate at which  $\lambda_{min}$  goes to zero, which, in turn, should be related to the rate at which a thermalized and gauge-fixed configuration approaches  $\partial \Omega$ .

#### **Relating** $\lambda_{min}$ **and Geometry**

It is generally accepted that

A. Cucchieri, T.M., PRD 2013

At very large volumes the functional integration gets concentrated on the boundary  $\partial \Omega$  of the first Gribov region  $\Omega$ .

But

The key point seems to be the rate at which  $\lambda_{min}$  goes to zero, which, in turn, should be related to the rate at which a thermalized and gauge-fixed configuration approaches  $\partial \Omega$ .

These are only qualitative statements!

How do we relate  $\lambda_1$ 

to the geometry of the Gribov region  $\Omega$ ?

## **The Region** $\Omega$ **: Properties**

Three important properties have been proven (D.Zwanziger, NPB 209, 1982) for the Gribov region  $\Omega$ :

- 1. the trivial vacuum  $A_{\mu} = 0$  belongs to  $\Omega$ ;
- 2. the region  $\Omega$  is convex;
- 3. the region  $\Omega$  is bounded in every direction.

(The same properties can be proven also for the fundamental modular region  $\Lambda$ )

The first property is trivial, since  $A_{\mu} = 0$  implies that  $\mathcal{M}(b, x; c, y)[0]$  is (minus) the Laplacian  $-\partial^2$  (which is a semi-positive-definite operator).

## **Convexity of** $\Omega$

The gauge condition  $\partial \cdot A = 0$  and the operators  $D^{bc}(x, y)[A]$ ,  $\mathcal{M}(b, x; c, y)[A] = -\partial^2 + \mathcal{K}[A]$  and  $\mathcal{K}[A]$  are linear in the gauge field  $A_{\mu}$ :

$$\mathcal{M}[\gamma A_1 + (1 - \gamma)A_2] = -\partial^2 + \mathcal{K}[\gamma A_1 + (1 - \gamma)A_2]$$
$$= \gamma \left(-\partial^2 + \mathcal{K}[A_1]\right) + (1 - \gamma)\left(-\partial^2 + \mathcal{K}[A_2]\right)$$
$$= \gamma \mathcal{M}[A_1] + (1 - \gamma)\mathcal{M}[A_2]$$

and, for  $\gamma \in [0,1]$ ,  $\mathcal{M}[\gamma A_1 + (1-\gamma)A_2]$  is semi-positive definite if  $\mathcal{M}[A_1]$  and  $\mathcal{M}[A_2]$  are semi-positive definite. Also

$$\gamma \,\partial \cdot A_1 \,+\, (1-\gamma) \,\partial \cdot A_2 \,=\, 0$$

if  $\partial \cdot A_1 = \partial \cdot A_2 = 0$ .  $\implies$  The convex combination  $\gamma A_1 + (1 - \gamma)A_2$ belongs to  $\Omega$ , for any value of  $\gamma \in [0, 1]$ , if  $A_1, A_2 \in \Omega$ .

#### **Boundary of** $\Omega$

Using properties 1 and 2 and with  $A_1 = 0$ ,  $A_2 = A$ ,  $1 - \gamma = \rho$  we have

$$\mathcal{M}[\rho A] = -\partial^2 + \mathcal{K}[\rho A] = (1 - \rho) (-\partial^2) + \rho \mathcal{M}[A]$$

and, if  $A \in \Omega$ , then  $\rho A \in \Omega$  for any value of  $\rho \in [0, 1]$ .

Since the color indices of  $\mathcal{K}[A]$  are given by  $\mathcal{K}^{bc}[A] \sim f^{bce}A^e_{\mu}$ , we have that all the diagonal elements of  $\mathcal{K}[A]$  are zero  $\Longrightarrow$  the trace of the operator  $\mathcal{K}[A]$  is zero.

The operator  $\mathcal{K}_{xy}^{bc}[A]$  is real and symmetric (under simultaneous interchange of x with y and b with c) and its eigenvalues are real  $\Longrightarrow$  at least one of the eigenvalues of  $\mathcal{K}[A]$  is (real and) negative. If  $\phi_{neg}$  is the corresponding eigenvector, that for a sufficiently large (but finite) value of  $\rho > 1$  the scalar product ( $\phi_{neg}, \mathcal{M}[\rho A]\phi_{neg}$ ) must be negative  $\Longrightarrow \mathcal{M}[\rho A]$  is not semi-positive definite and  $\rho A \notin \Omega$ .

## **The Infinite-Volume Limit**

In order to study the infrared sector of the theory on the lattice we need to remove the infrared cutoff  $\implies$  take the infinite-volume limit.

At very large volumes the functional integration gets concentrated on the boundary  $\partial \Omega$  of the first Gribov region  $\Omega$ .

For very large dimensionality and for large volumes, by considering the interplay among the volume of the configuration space, the Boltzmann weight and the step function used to constrain the functional integration to  $\Omega$ , one expects that entropy favors configurations near the boundary  $\partial\Omega$ .  $\Rightarrow$  As said above: now relate  $\lambda_1$  to geometry of  $\Omega$ 

## Lower bound for $\lambda_1$ (I)

Consider a configuration A' belonging to the boundary  $\partial \Omega$  of  $\Omega$  and write

$$\lambda_1 \left[ \mathcal{M}[\rho A'] \right] = \lambda_1 \left[ (1-\rho) \left( -\partial^2 \right) + \rho \mathcal{M}[A'] \right]$$

From the second property,  $\rho A' \in \Omega$  for  $\rho \in [0, 1]$ . Since

$$\frac{\lambda_1}{\left[\left(1-\rho\right)\left(-\partial^2\right)\,+\,\rho\,\mathcal{M}[A']\right]}$$

$$= \min_{\chi} \left( \chi, \left[ \left( 1 - \rho \right) \left( -\partial^2 \right) + \rho \mathcal{M}[A'] \right] \chi \right) ,$$

with  $(\chi, \chi) = 1$  and  $\chi \neq$  constant, we can use the concavity of the minimum function

$$\min_{\chi} \left( \chi, \left[ M_1 + M_2 \right] \chi \right) \ge \min_{\chi} \left( \chi, M_1 \chi \right) + \min_{\chi} \left( \chi, M_2 \chi \right)$$

QCD-TNT 4

## Lower bound for $\lambda_1$ (II)

We find

$$\begin{split} \lambda_1 \left[ \mathcal{M}[\rho A'] \right] &= \lambda_1 \left[ (1-\rho) \left( -\partial^2 \right) + \rho \mathcal{M}[A'] \right] \\ &\geq (1-\rho) \min_{\chi} \left( \chi, \left( -\partial^2 \right) \chi \right) + \rho \min_{\chi} \left( \chi, \mathcal{M}[A'] \chi \right) \\ &= \left( 1-\rho \right) p_{min}^2 \,. \end{split}$$

Recall that  $A' \in \partial \Omega \implies$  the smallest non-trivial eigenvalue of the FP matrix  $\mathcal{M}[A']$  is null, and that the smallest non-trivial eigenvalue of (minus) the Laplacian  $-\partial^2$  is  $p_{min}^2$ .

In the Abelian case one has  $\mathcal{M} = -\partial^2$  and  $\lambda_1 = p_{min}^2$ .  $\implies$ All non-Abelian effects are included in the  $(1 - \rho)$  factor (and in the inequality).

## Lower bound for $\lambda_1$ (III)

As the lattice side *L* goes to infinity,  $\lambda_1 [\mathcal{M}[\rho A']]$  cannot go to zero faster than  $(1 - \rho) p_{min}^2$ . Since  $p_{min}^2 \sim 1/L^2$  at large  $L \Longrightarrow \lambda_1$  behaves as  $L^{-2-\alpha}$  in the same limit, with  $\alpha > 0$ , only if  $1 - \rho$  goes to zero at least as fast as  $L^{-\alpha}$ .

With  $\rho A' = A$  the above inequality may also be written as

$$\lambda_1 \left[ \mathcal{M}[A] \right] \geq \left[ 1 - \rho(A) \right] p_{min}^2$$

Here  $1 - \rho(A) \leq 1$  measures the distance of a configuration  $A \in \Omega$  from the boundary  $\partial \Omega$  (in such a way that  $\rho^{-1}A \in \partial \Omega$ ). This result applies to any Gribov copy belonging to  $\Omega$ .

# **Summarizing**

Using properties of  $\Omega$  and the concavity of the minimum function, one can show (A. Cucchieri, TM, PRD 2013)

 $\lambda_{min} \left[ \mathcal{M}[A] \right] \geq \left[ 1 - \rho(A) \right] p_{min}^2$ 

Here  $1 - \rho(A) \leq 1$  measures the distance of a configuration  $A \in \Omega$ from the boundary  $\partial \Omega$  (in such a way that  $\rho^{-1}A \equiv A' \in \partial \Omega$ ). This result applies to any Gribov copy belonging to  $\Omega$ .

Recall that  $A' \in \partial \Omega \implies$  the smallest non-trivial eigenvalue of the FP matrix  $\mathcal{M}[A']$  is null, and that the smallest non-trivial eigenvalue of (minus) the Laplacian  $-\partial^2$  is  $p_{min}^2$ .

In the Abelian case one has  $\mathcal{M} = -\partial^2$  and  $\lambda_{min} = p_{min}^2$  $\implies$  non-Abelian effects are included in the  $(1 - \rho)$  factor

## **Simulating the Math**

We used 70 configurations, for the SU(2) case at  $\beta = 2.2$ , for  $V = 16^4$ ,  $24^4, 32^4, 40^4$  and 50 configurations for  $V = 48^4, 56^4, 64^4, 72^4, 80^4$ .

In order to verify the third property of the region  $\Omega$  we applied scale transformations  $\widehat{A}^{(i)}_{\mu}(x) = \tau_i A^{(i-1)}_{\mu}(x)$  to the gauge configuration A with

$$\tau_0 = 1,$$

$$\delta = 1.001$$
 if  $\lambda_1 \geq 5 imes 10^{-3}$ ,

$$\delta = 1.0005 \text{ if } \lambda_1 \in [5 \times 10^{-4}, 5 \times 10^{-3})$$

and  $\delta = 1.0001$  if  $\lambda_1 < 5 \times 10^{-4}$ ,

where  $\lambda_1$  is evaluated at the step i - 1. After *n* steps, the modified gauge field  $\widehat{A}_{\mu}^{(n)}(x)$  does not belong anymore to the region  $\Omega$ , i.e. the eigenvalue  $\lambda_1$  of  $\mathcal{M}[\widehat{A}^{(n)}]$  is negative (while  $\lambda_2$  is still positive).

## **Crossing the Horizon (I)**

N	$\max(n)$	$\min(n)$	$\langle n  angle$	$R_{ m before}$	$R_{ m after}$
16	30	6	17.2	15(3)	-30(12)
24	27	4	15.1	20(7)	-26(6)
32	19	5	11.7	26(9)	-51(20)
40	18	4	9.4	155(143)	-21(6)
48	13	2	7.8	21(5)	-21(5)
56	12	3	7.6	16(4)	-21(7)
64	11	2	6.8	20(7)	-42(18)
72	11	2	6.1	129(96)	-42(13)
80	12	3	6.1	15(4)	-24(4)

The maximum, minimum and average number of steps n, necessary to "cross the Gribov horizon" along the direction  $A_{\mu}^{b}(x)$ , as a function of the lattice size N. We also show the ratio  $R[A] = (S''')^2/(S''S'''')$ , divided by 1000, for the modified gauge fields  $\tau_{n-1}A_{\mu}^{b}(x)$  and  $\tau_{n}A_{\mu}^{b}(x)$ , i.e. for the configurations immediately before and after crossing  $\partial\Omega$ .

#### **Crossing the Horizon (II)**

#### The case of a typical configuration.





Plot of the ratio R, as a function of the iteration step i, for a configuration with lattice volume  $16^4$ . Plot of  $\lambda_2$  (full circes),  $|\mathcal{E}'''|$ (full squares) and  $\mathcal{E}''''$  (full triangles) as a function of the iteration step *i*, for the same configuration.

#### **Crossing the Horizon (III)**

#### The case $R \approx 0$ (configuration on $\partial \Omega \cap \partial \Lambda$ ).





Plot of the ratio R, as a function of the iteration step i, for a configuration with lattice volume  $48^4$ . Plot of  $\lambda_2$  (full circes),  $|\mathcal{E}'''|$ (full squares) and  $\mathcal{E}''''$  (full triangles) as a function of the iteration step *i*, for the same configuration.

# **New Inequality**



Using  $A' = \tilde{\tau} A \equiv A(\tau_{n-1} + \tau_n)/2 \in \partial \Omega$ and  $\rho = 1/\tilde{\tau} < 1$ :

plot of the inverse of the (previous) lower bound (empty circles), of  $1/G(p_{min})$  (full triangles), of  $\lambda_1$  (full squares) and of the quantity  $(1 - \rho) p_{min}^2$  (full circles) as a function of the inverse lattice size 1/N.

The new inequality  $\lambda_1 [\mathcal{M}[A]] \ge [1 - \rho(A)] p_{min}^2$  becomes an equality if and only if the eigenvectors corresponding to the smallest nonzero eigenvalues of  $\mathcal{M}[A]$  and  $-\partial^2$  coincide.  $\Longrightarrow$ The eigenvector  $\psi_{min}$  is very different from the plane waves corresponding to  $p_{min}$ .

These results explain the non-enhancement of G(p) in the IR.

We've ventured outside the region Ω (away from sampled configurations)

- We've ventured outside the region Ω (away from sampled configurations)
- Comparison of measurements for non-representative configurations to usual ones allows test of new bounds

- We've ventured outside the region Ω (away from sampled configurations)
- Comparison of measurements for non-representative configurations to usual ones allows test of new bounds
- Results suggest that all non-perturbative features of a minimal-Landau-gauge configuration A ∈ Ω are related to its normalized distance ρ from the "origin" A = 0 or, equivalently, to its normalized distance 1 ρ from the boundary ∂Ω

- We've ventured outside the region Ω (away from sampled configurations)
- Comparison of measurements for non-representative configurations to usual ones allows test of new bounds
- Results suggest that all non-perturbative features of a minimal-Landau-gauge configuration A ∈ Ω are related to its normalized distance ρ from the "origin" A = 0 or, equivalently, to its normalized distance 1 ρ from the boundary ∂Ω
- We now begin to understand why no ghost enhancement (scaling solution) is seen on the lattice