Introduction to Quantum Walks – part 2

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4-18 June 2024

DTQW vs DTRW – 1D

Recall DTQW:

- System: $|x, c\rangle$
- Coin tossing operation:

$$C|\rightarrow\rangle = \alpha|\rightarrow\rangle + \beta|\leftarrow\rangle$$

$$C|\leftarrow\rangle = \beta^* e^{i\theta}|\rightarrow\rangle - \alpha^* e^{i\theta}|\leftarrow\rangle$$

- Shift operation:

$$S|x, \rightarrow\rangle = |x+1, \rightarrow\rangle$$

$$S|x, \leftarrow\rangle = |x-1, \leftarrow\rangle$$

- One step operator:

$$U = S(I \otimes C)$$

$$U|\psi(t)\rangle = |\psi(t+1)\rangle$$

DTQW vs DTRW – 1D

- Parametrisation of the coin operator

$$C = e^{i\theta \vec{n}\cdot\vec{\sigma}}$$

  Rotation of qubit's Bloch vector about $\vec{n}$-axis by angle 2θ (3 parameters)

- Hadamard coin: $\theta = \pi/2$, $\vec{n} = (1,0,1)/\sqrt{2}$
- We often use $\vec{n} = (0,1,0)$, therefore the coin is parametrised only by θ:

$$C = e^{i\theta\sigma_y} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

DTQW vs DTRW – 1D

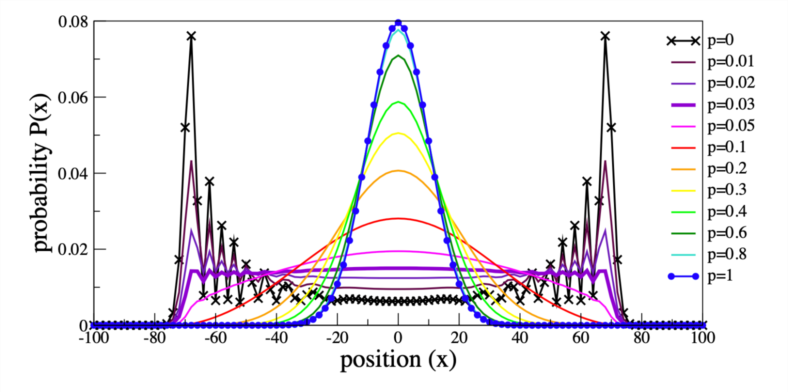
Differences between DTRW and DTQW:

- See video
- Probability distribution: Gaussian vs anti-Gaussian
- Standard deviation: $O(\sqrt{t})$ vs $O(t)$
- Diffusive vs Ballistic
- Potential source for quantum speedup (as we will see later)

DTQW vs DTRW – 1D

How to get DTRW from DTQW?

Decoherence – we measure the coin or the walker!



V. Kendon, Math. Struct. in Comp. Sci 17(6) pp 1169-1220 (2006)

DTQW 1D – properties

Properties of the evolution operator $U = S(I \otimes C)$

- Translational invariance
- Plane waves:

$$|k\rangle = \sum_{x=-\infty}^{+\infty} e^{ikx}|x\rangle, \qquad k \in [-\pi, \pi)$$

- Action of the shift operator on plane waves

$$S|k,\rightarrow\rangle = \sum_{x=-\infty}^{+\infty} e^{ikx}|x+1,\rightarrow\rangle = e^{-ik} \sum_{x=-\infty}^{+\infty} e^{ikx}|x,\rightarrow\rangle = e^{-ik}|k,\rightarrow\rangle$$

$$S|k,\leftarrow\rangle = \sum_{x=-\infty}^{+\infty} e^{ikx}|x-1,\leftarrow\rangle = e^{ik} \sum_{x=-\infty}^{+\infty} e^{ikx}|x,\leftarrow\rangle = e^{-ik}|k,\leftarrow\rangle$$

DTQW 1D – properties

- Eigenvectors

$$|\lambda_{k,c}\rangle = |k\rangle \otimes \begin{pmatrix} \alpha_{k,c} \\ \beta_{k,c} \end{pmatrix}$$

- Eigenvalue problem

$$U|\lambda_{k,c}\rangle = S(I \otimes C)|\lambda_{k,c}\rangle = \lambda_{k,c}|\lambda_{k,c}\rangle$$

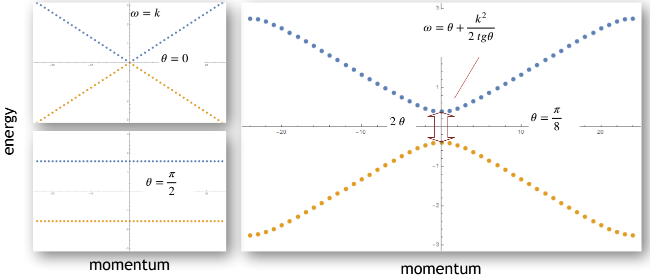
- Using translational invariance

$$\begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_{k,c} \\ \beta_{k,c} \end{pmatrix} = \lambda_{k,c} \begin{pmatrix} \alpha_{k,c} \\ \beta_{k,c} \end{pmatrix}$$

DTQW 1D – properties

Eigenvalues:

$$\lambda_{k,\pm} = e^{\pm i\omega_k} = \cos\theta\cos k \pm i\sqrt{1-\cos^2\theta\cos^2 k}$$



DTQW 1D – properties

- Two bands: $c = \pm$
- Symmetry: $\lambda_{k,+} = \lambda^*_{k,-}$
- Quasi energies:

$$\pm\omega_k \in [-\pi, \pi)$$

- Can we find an analogous physical system?
- What is the Hamiltonian H of the system?

$$U = S(I \otimes C) = e^{-iH\Delta t}$$

DTQW 1D – physical analogue

- How to implement shift operation?

$$S|x,\rightarrow = |x+1,\rightarrow\rangle, \qquad S|x,\leftarrow = |x-1,\leftarrow\rangle$$

- Recall translation operator:

$$e^{ip\Delta x}|x\rangle = |x+\Delta x\rangle$$

- Add conditioning (von Neumann measurement model)

$$A|a_i\rangle = a_i|a_i\rangle, \qquad e^{ipA}|x,a_i\rangle = |x+a_i,a_i\rangle$$

- For DTQW $A = \sigma_z$

Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A, 48, 1687 (1993)

$$S = e^{ip\sigma_z}$$

DTQW 1D – physical analogue

- Evolution operator:

$$U = S(I \otimes C) = e^{ip\sigma_z} e^{i\theta\sigma_y}$$

- Let v be a particle's velocity and ω a coin's angular velocity
- Introduce: unit step $1 = -v\Delta t$ and rotation angle $\theta = -\omega\Delta t$

$$U = S(I \otimes C) = e^{-ivp\sigma_z \delta t} e^{-i\omega\sigma_y \Delta t}$$

- Trotterisation: $\Delta t = \frac{t}{n} \ll 1$

$$U^n = \left(e^{-ivp\sigma_z \frac{t}{n}} e^{-i\omega\sigma_y \frac{t}{n}}\right)^n \approx e^{-i(vp\sigma_z + \omega\sigma_y)t}$$

F.W. Strauch, Phys. Rev. A, 73, 054302 (2006)

DTQW 1D – Dirac Hamiltonian

- DTQW effective Hamiltonian:

$$H_{DTQW} = vp\sigma_z + \omega\sigma_y$$

- Dirac Hamiltonian:

$$H_D = cp\sigma_z + mc^2\sigma_y$$

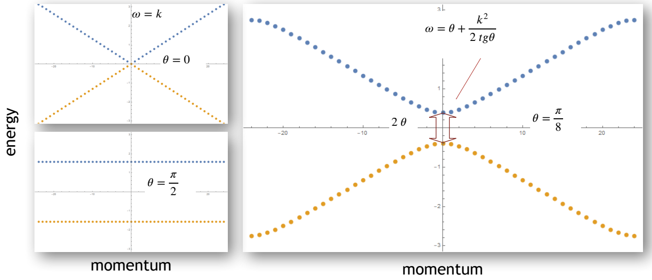
$$H_D^2 = E^2 = c^2p^2\sigma_z^2 + m^2c^4\sigma_y^2 + pmc^3(\sigma_z\sigma_y + \sigma_y\sigma_z)$$
$$E^2 = c^2p^2 + m^2c^4$$

- DTQW describes a relativistic free particle in a discrete space-time! (Feynman's check-board model)

$$v = c, \quad \omega = mc^2$$

DTQW – Dirac analogies

- See animations: *http : //home.pcisys.net/ bestwork.1/QQM/dimension$_d$irac.htm*
- Free particle: constant velocity ≡ ballistic
- Coin: doubling of states – two quasi-energies (particles/antiparticles)
- Coin tossing operator is an analogue of mass (Dirac energy gap is $2mc^2$)



DTQW – Dirac analogies

- Example 1: $\theta = 0$ (massless particle)

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|\psi(0)\rangle = |x, \rightarrow\rangle, \ |\psi(1)\rangle = |x{+}1, \rightarrow\rangle, \ \ldots, |\psi(t)\rangle = |x{+}t, \rightarrow\rangle$$

- Example 2: $\theta = \pi/2$ (infinitely massive particle)

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$|\psi(0)\rangle = |x, \rightarrow\rangle, \quad |\psi(1)\rangle = -|x-1, \leftarrow\rangle,$$
$$|\psi(2)\rangle = -|x, \rightarrow\rangle, \quad |\psi(3)\rangle = -|x-1, \leftarrow\rangle,$$

DTQW – Dirac analogies

- Dirac velocity:

$$v = \frac{dx}{dt} = i[H, x] = c\sigma_z$$

- There are only two eigenvalues of velocity operator: $\pm c$!
- Particle moves in one direction with the speed of light, but turns back due to mass – Zitterbewegung (trembling motion)
- The heavier the particle, the more often it turns back (extremely fast oscillations, so we only see average motion which is $\ll c$)

DTQW – Dirac analogies

- DTQW velocity:

$$v = x(t+1) - x(t)$$

$$x = \sum_{x=-\infty}^{+\infty} |x\rangle\langle x| \otimes I$$

$$v = UxU^\dagger - x = I \otimes \sigma_z$$

- There are only two eigenvalues of velocity operator: ± 1

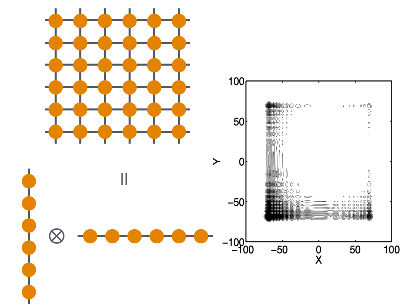
DTQW – 2D

Quantum walks in 2D

- State: $|x, y, c\rangle$
- Coin: $c = \rightarrow, \uparrow, \leftarrow, \downarrow$
- Simplest generalisation:

$$U_{2D} = U_{1D} \otimes U_{1D} = [S_x(I \otimes C_{\rightarrow,\leftarrow})] \otimes [S_y(I \otimes C_{\uparrow,\downarrow})]$$

T. D. Mackay, S. D. Bartlett, L. T. Stephanson and B. C. Sanders, J. Phys. A: Math. Gen. 35, 2745 (2002)



DTQW – 2D

Quantum walks in 2D – general coin

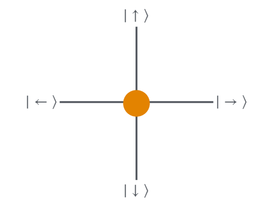
- State: $|x, y, c\rangle$
- Shift operator:

$$S|x, y, \rightarrow\rangle = |x+1, y, \rightarrow\rangle, \quad S|x, y, \uparrow\rangle = |x, y+1, \uparrow\rangle$$

$$S|x, y, \leftarrow\rangle = |x-1, y, \leftarrow\rangle, \quad S|x, y, \downarrow\rangle = |x, y-1, \downarrow\rangle$$

- Coin tossing operator: C – arbitrary 4×4 unitary matrix
- Evolution: $U = S(I \otimes C)$

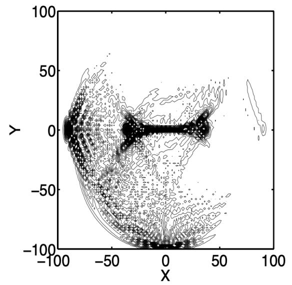
T. D. Mackay, S. D. Bartlett, L. T. Stephanson and B. C. Sanders, J. Phys. A: Math. Gen. 35, 2745 (2002)



DTQW – 2D

Example 1: C – Discrete Fourier'e Transform

$$C_{DFT} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix}, \quad \omega = e^{i2\pi/4} = i$$

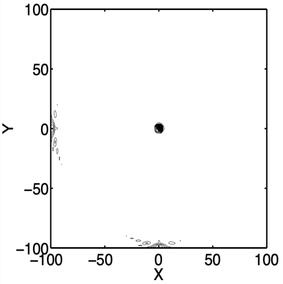


T. D. Mackay, S. D. Bartlett, L. T. Stephanson and B. C. Sanders, J. Phys. A: Math. Gen. 35, 2745 (2002)

DTQW – 2D

Example 2: C – Grover operator

$$C_G = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = 2|\psi\rangle\langle\psi| - I, \qquad |\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$



T. D. Mackay, S. D. Bartlett, L. T. Stephanson and B. C. Sanders, J. Phys. A: Math. Gen. 35, 2745 (2002)

DTQW – 2D

Grover walk – new type of localisation

$$\bar{p}_{\infty}(x_0) > \frac{1}{2}$$

- System remembers its initial position
- Usual localisation – lack of translational symmetry (e.g. Anderson)
- Grover walk is translationally symmetric – what happens?
- Extreme degeneracy: more than 25% of eigenvalues are -1, other eigenvalues are also heavily degenerated

N. Inui, Y. Konishi, and N. Konno, Phys. Rev. A, 69:052323, (2004)

DTQW – beyond 2D

Walks on general d-dimensional lattices

- State: $|x_1, x_2, \ldots, x_d, c\rangle$
- Coin 1: $c = 1, 2, \ldots, 2d$
- Shift 1:

$$S|x_1, x_2, \ldots, x_j, \ldots, x_d, 2j\rangle = |x_1, x_2, \ldots, x_j + 1, \ldots, x_d, 2j\rangle$$

- Coin 2: $c = 0, 1, 2, \ldots, 2^d - 1$ (e.g. binary $011\ldots0$)
- Shift 2:

$$S|x_1, x_2, x_3, \ldots, x_d, 011\ldots0\rangle =$$

$$|x_1 + 1, x_2 - 1, x_3 - 1, \ldots, x_d + 1, 011\ldots0\rangle$$

DTQW – beyond 2D

Walks on general *d*-dimensional lattices with 2D coin

- State: $|x_1, x_2, \ldots, x_d, c\rangle$
- Coin 3: $c = \leftarrow, \rightarrow$
- Evolution:

$$U = S_{x_1}(I \otimes C_1) S_{x_2}(I \otimes C_2) \ldots S_{x_d}(I \otimes C_d)$$

- Example: 2D Dirac walk

$$U = e^{ip_x\sigma_x} e^{i\theta\sigma_y} e^{ip_z\sigma_z} e^{i\theta\sigma_y}$$

Continuous version:

$$U = e^{-iHt}, \qquad H = vp_x\sigma_x + vp_z\sigma_z + \omega\sigma_y$$

DTQW – beyond 2D

Walks without a coin – staggered QW

R. Portugal, R. A. M. Santos, T. D. Fernandes, and D. N. Conclaves, Quant. Inf. Proc., 15: 85–101, (2016)

- State: $|x\rangle$
- Evolution:

$$U = U_0 U_1$$

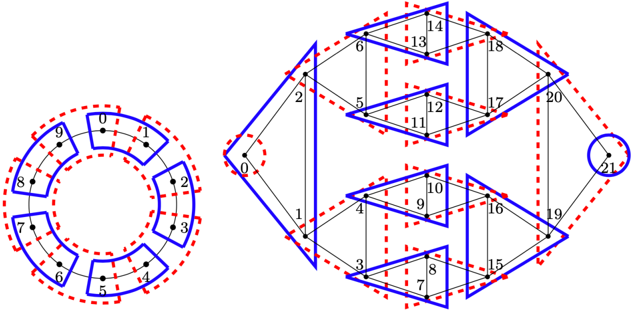
$$U_0 = 2\sum_{k=0}^{m-1} |\alpha_k\rangle\langle\alpha_k| - I, \qquad U_1 = 2\sum_{k=0}^{n-1} |\beta_k\rangle\langle\beta_k| - I$$

$$|\alpha_k\rangle = \sum_{x\in\alpha_k} a_{x,k}|x\rangle, \qquad |\beta_k\rangle = \sum_{x\in\beta_k} b_{x,k}|x\rangle$$

$$\langle\alpha_k|\alpha_{k'}\rangle = \delta_{k,k'}, \qquad \langle\beta_k|\beta_{k'}\rangle = \delta_{k,k'}$$

DTQW – beyond 2D

Example tesselations

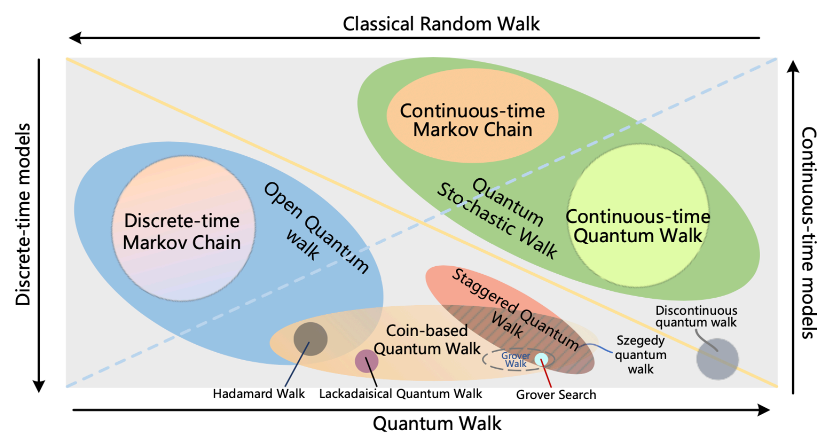


(a) Cycle with 2-site polygons

(b) Glued triangle tree

R. Portugal, R. A. M. Santos, T. D. Fernandes, and D. N. Conclaves, Quant. Inf. Proc., 15: 85–101, (2016)

QW overview



Xiaogang Qiang, Shixin Ma, and Haijing Song arXiv:2404.04178 (2024)