Boost Invariance and Stability for Relativistic Dissipative Hydrodynamical Theories with Relaxation

Hugo Marrochio, David Fogaça, Jorge Noronha

February 2014 - Retinha XXV

Introduction

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- Heavy-ion collisions at LHC

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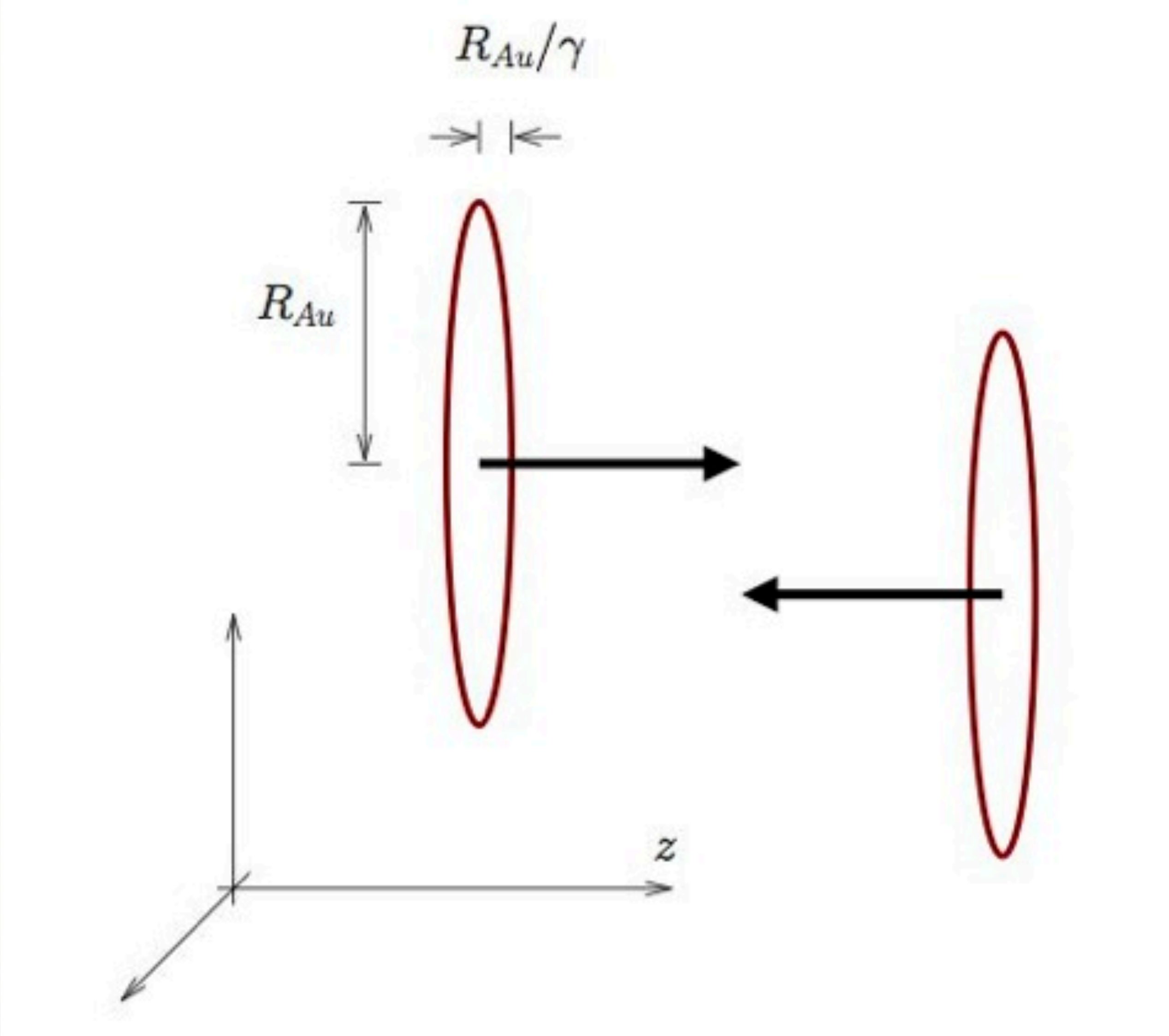
- Problems with Navier-Stokes



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- Thermodynamics of conformal field theory - good approximation

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Navier-Stokes (NS)

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BRSSS

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Navier-Stokes (NS) BRSSS Israel-Stewart (IS)

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- Shear Viscosity η and relaxation-like term τ
- Dissipative Tensor - second order in gradients of velocity
- Recovers NS for τ=0
- τ calculated with holography and gradient expansion

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Israel-Stewart (IS)

- For some values of τ no known problems with causality and stability
- Shear Viscosity η and relaxation term τ
- Dissipative Tensor - Dynamical degree of freedom - Part of a Differential Equation
- Recovers NS for $\tau=0$
- Theory is consistent with Boltzmann Eq.
- Used in numerical simulations

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Relativistic Dissipative Fluid Dynamics

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Energy $$DT + \frac{\theta T}{3} + \frac{\pi_{\mu\nu}\sigma^{\mu\nu}}{3s} = 0$$

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Relativistic Dissipative Fluid Dynamics

$$g_{\mu\nu} = \mathrm{diag}(-,+,+,+)$$

$$D = u^\mu \partial_\mu$$

$$\theta = \partial_\mu u^\mu$$

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$\Delta^{\mu\nu\alpha\beta} = \frac{(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha})}{2} - \frac{\Delta^{\mu\nu}\Delta^{\alpha\beta}}{3}$$

$$A^{<\mu\nu>} = \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$$

$$\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$

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- Linear perturbation in the x direction

$$u^{\mu} = (1, 0, 0, 0) + (0, \delta u^{x}(t, x), 0, 0)$$

$$T = T_0 + \delta T(t, x)$$

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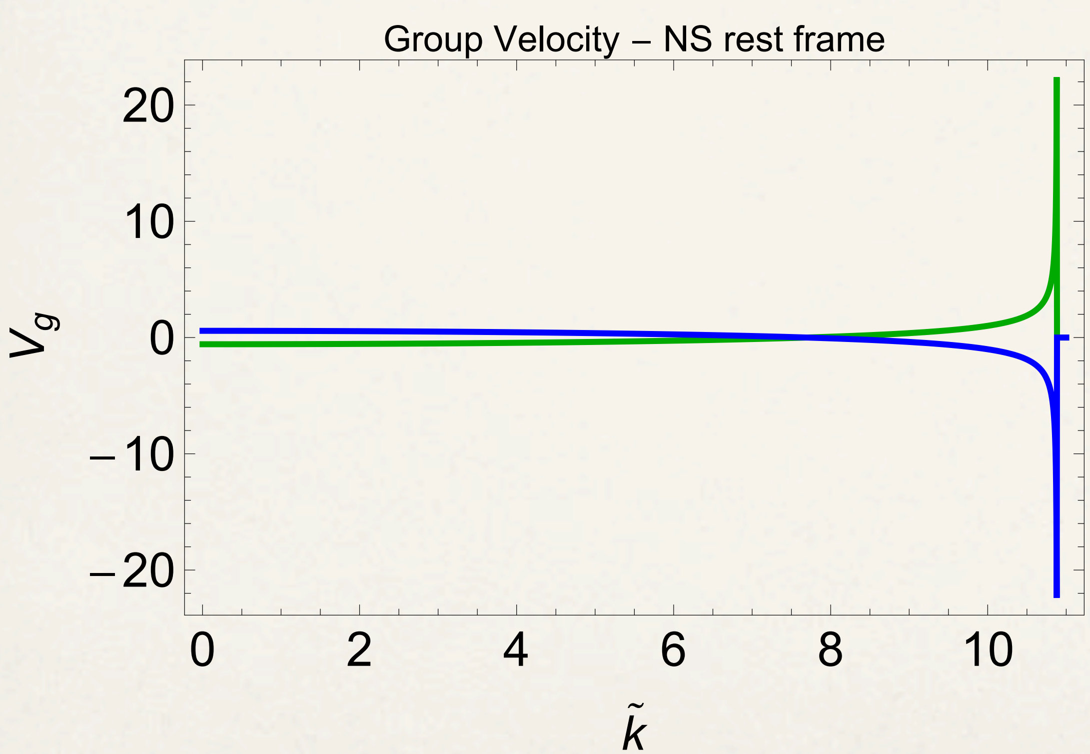

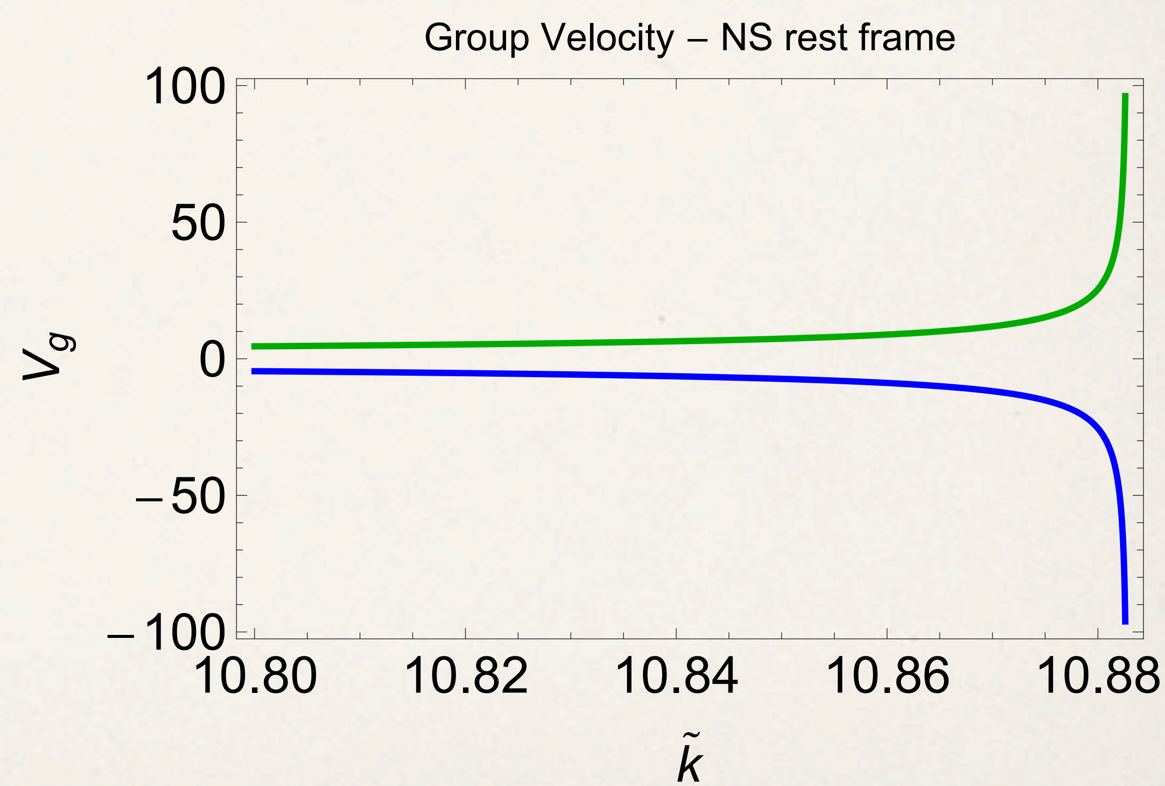

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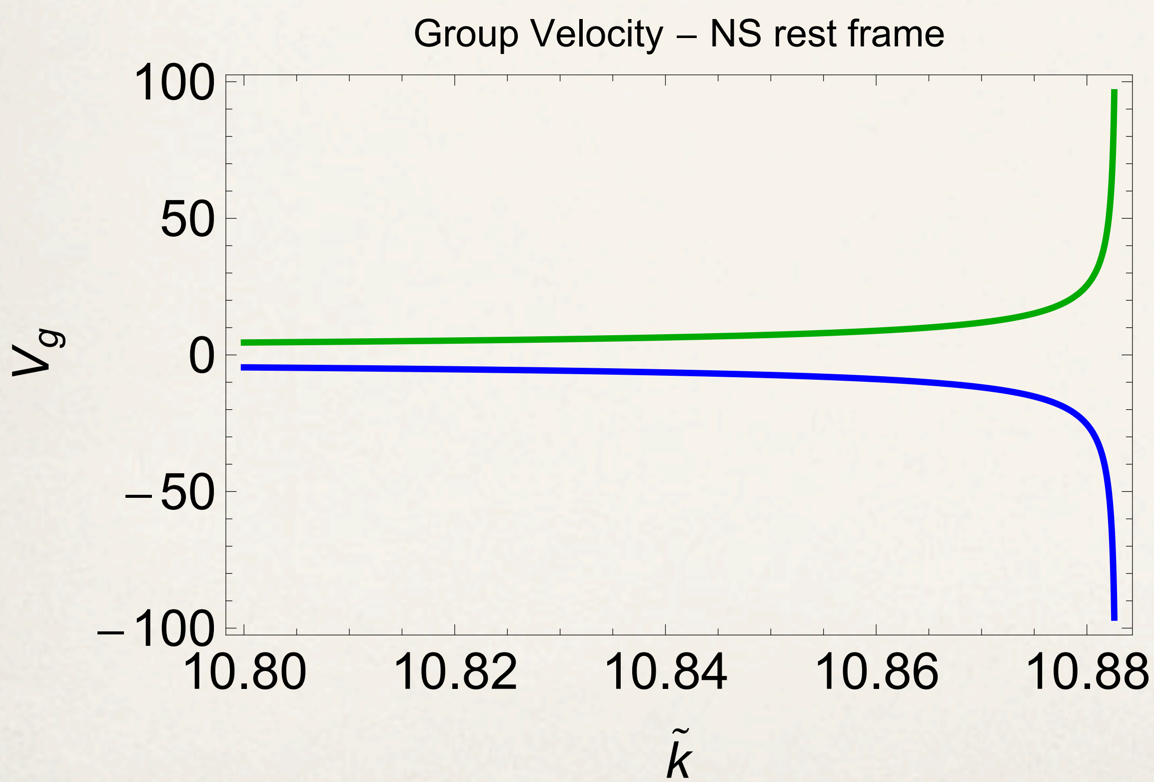

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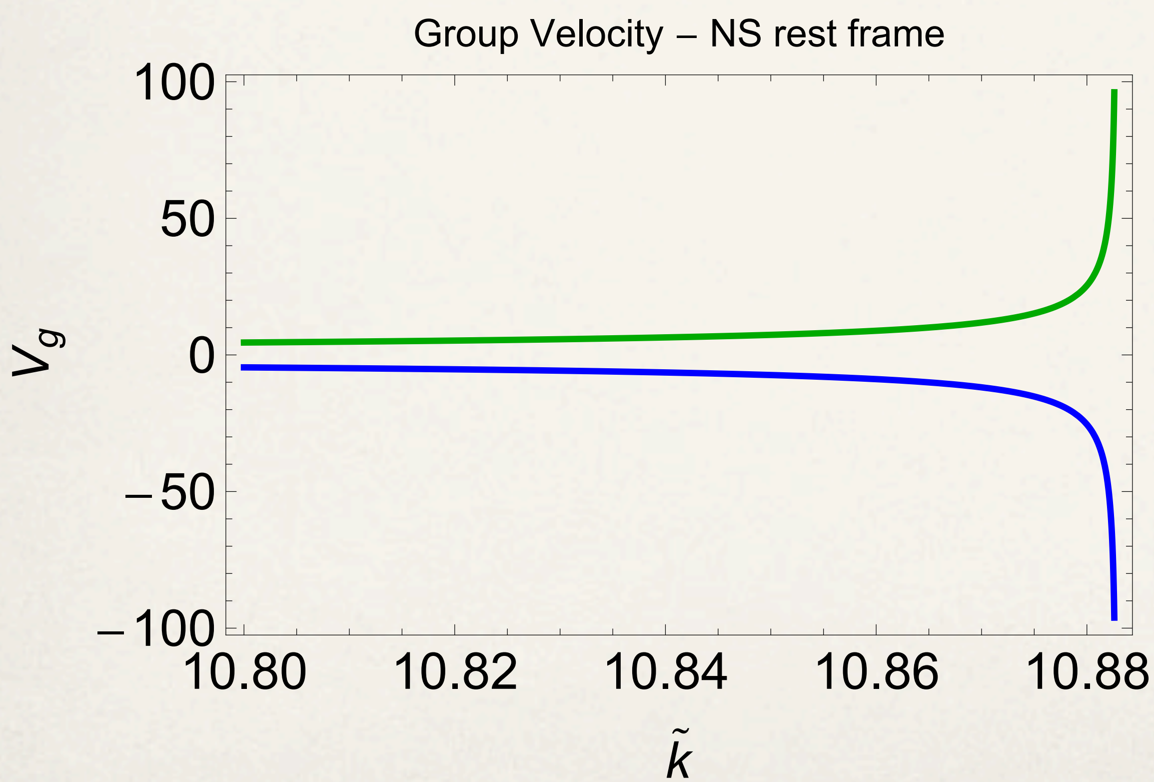

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Gradient Expansion

Calculate coefficients through holography

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- Simplest Israel-Stewart Equation: First order in gradients of 4-velocity:

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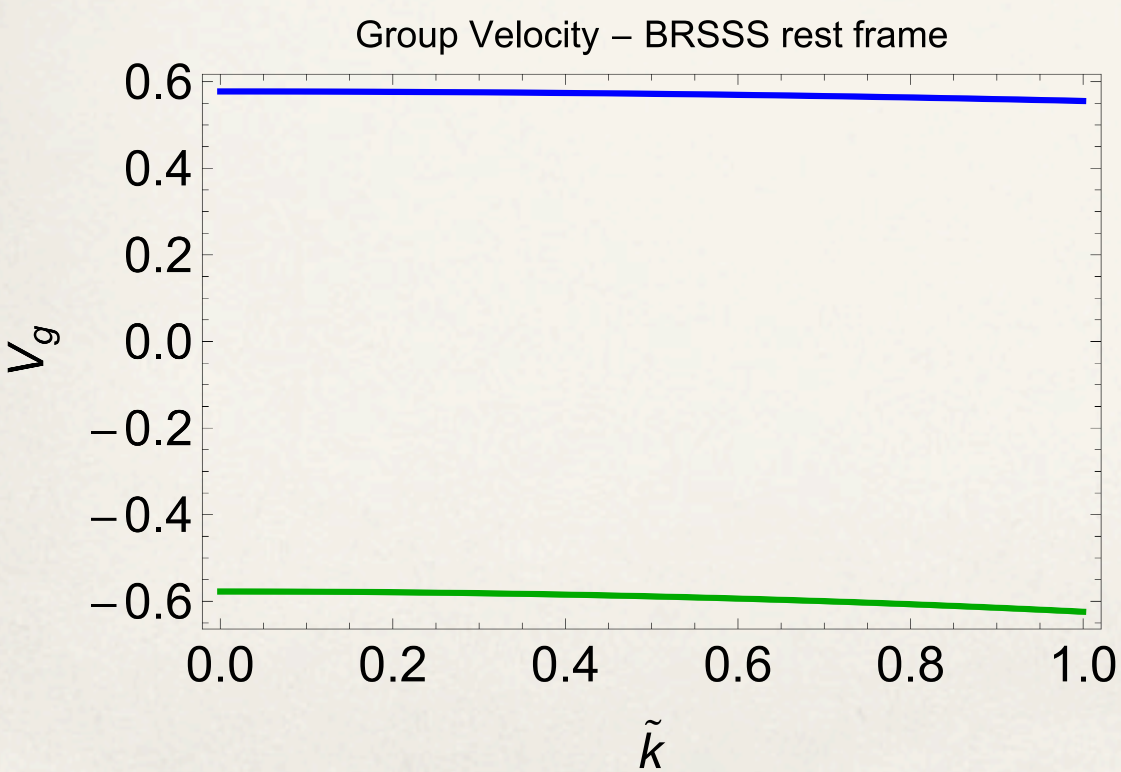

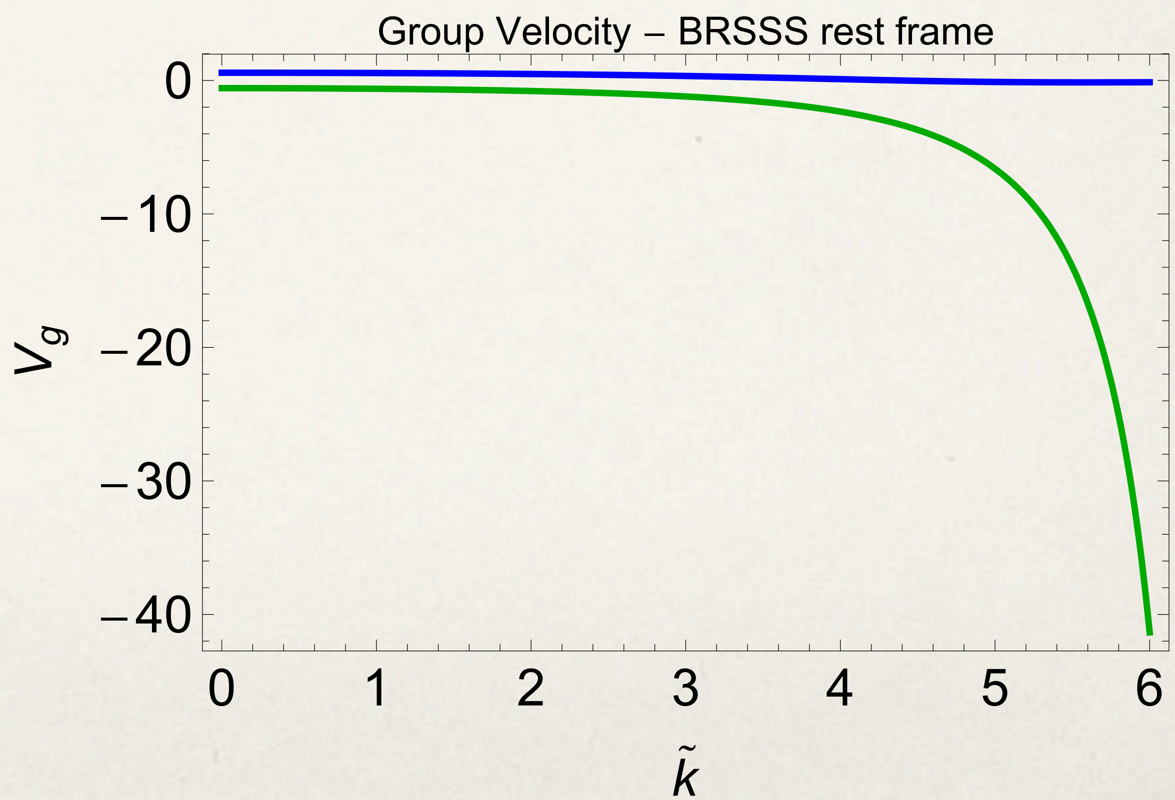

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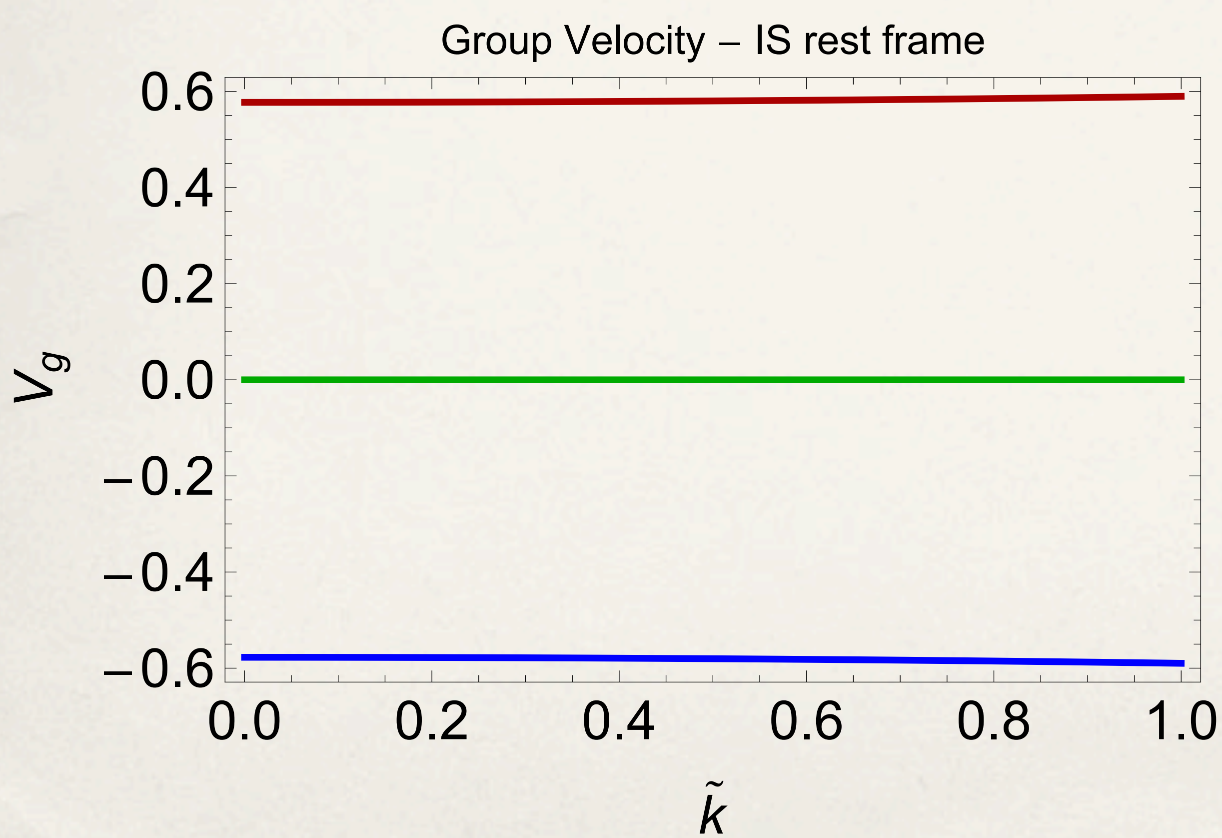

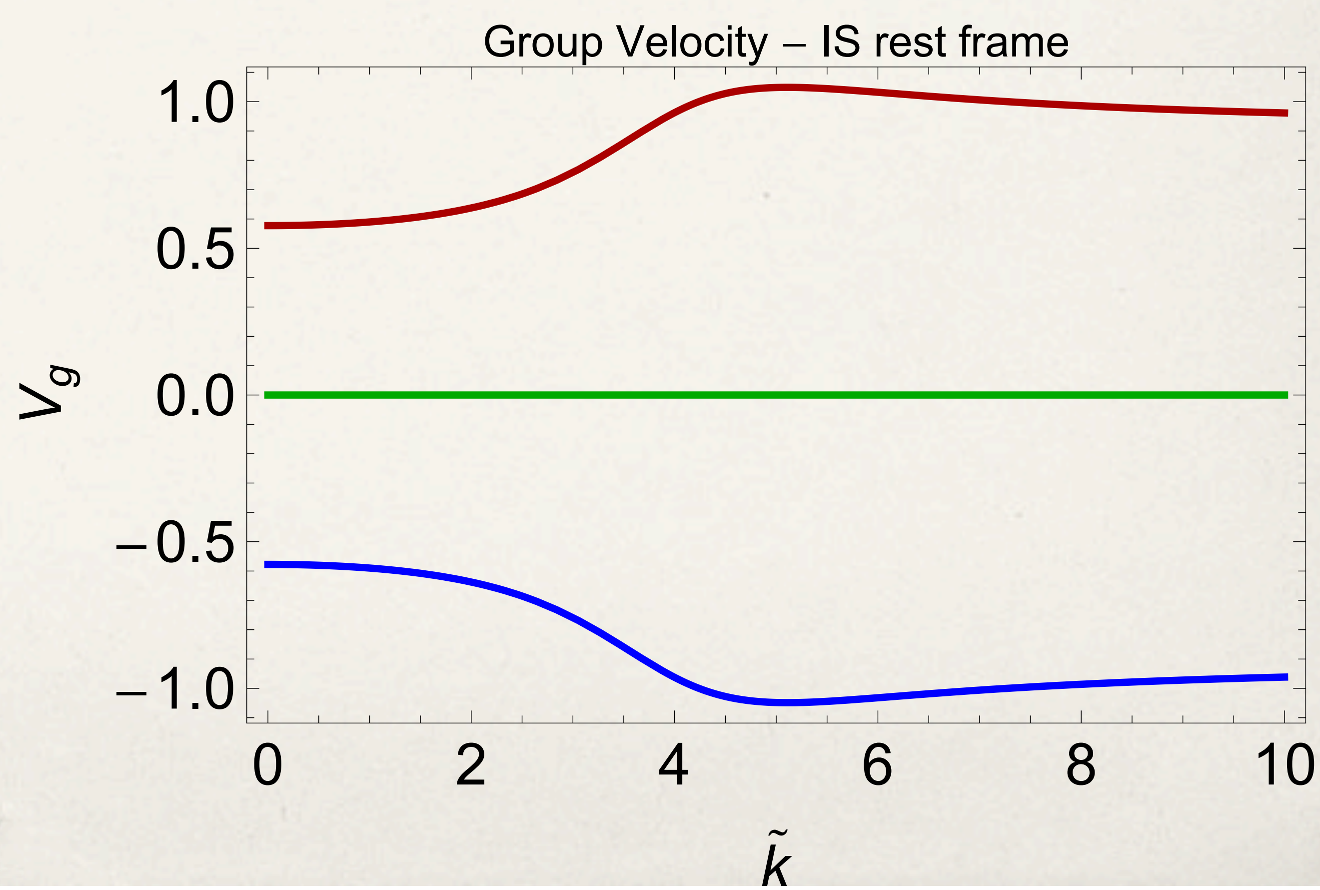

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- For solutions of the type: $$\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$ $$\omega = Re[\omega] + \mathrm{i}\ Im[\omega]$$

- The solution is unstable if ω has a positive imaginary value

$$\mathrm{e}^{Im[\omega]t}\mathrm{e}^{\mathrm{i}(kx-Re[\omega]t)}$$

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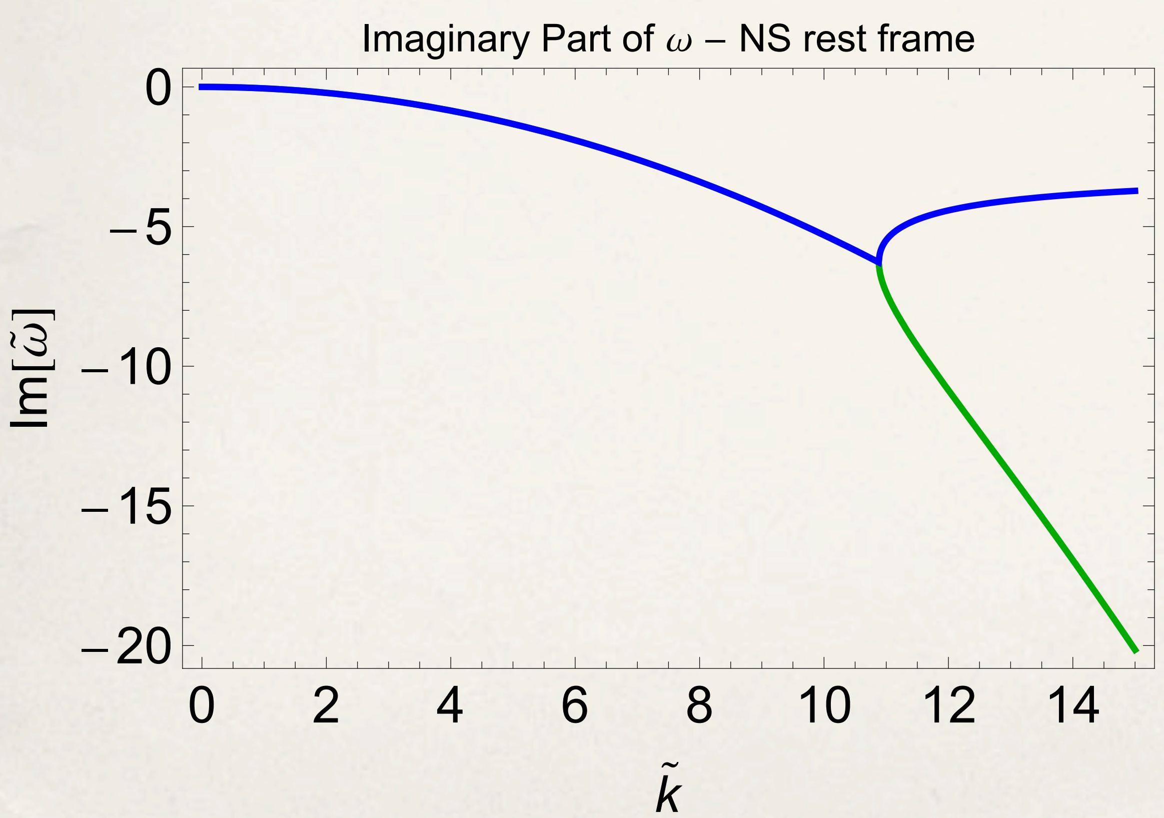

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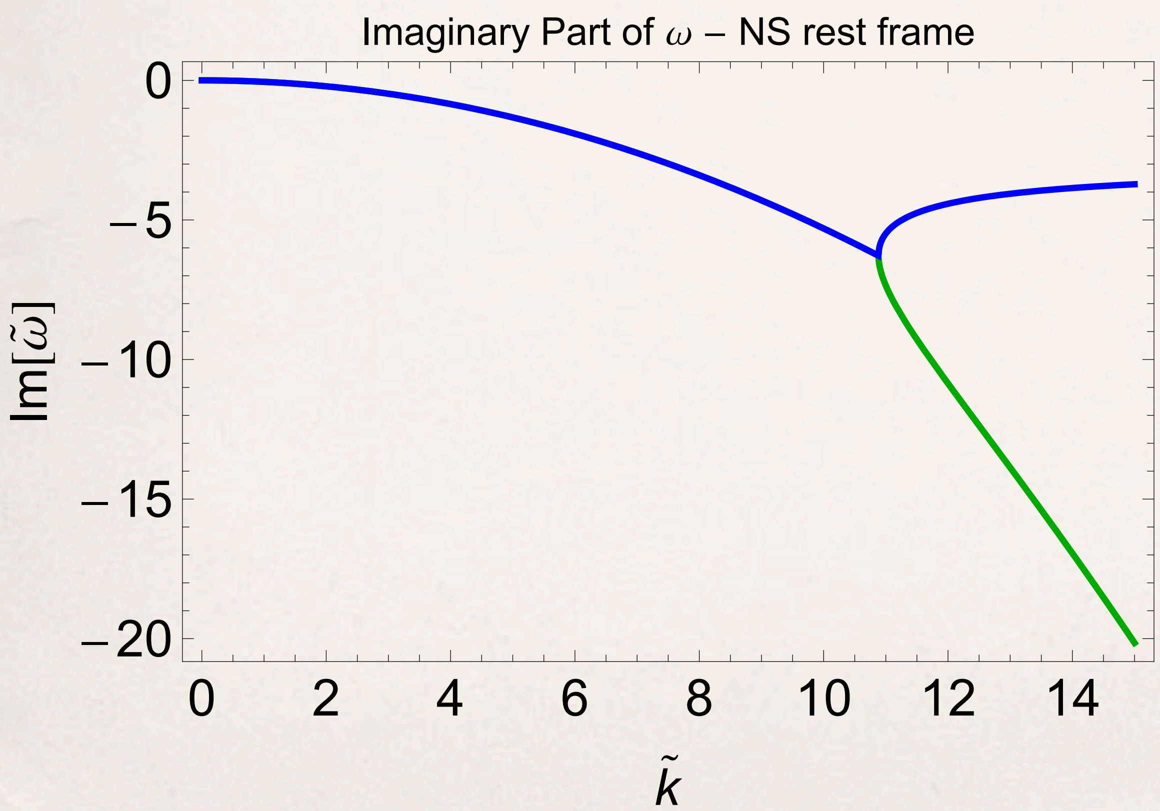

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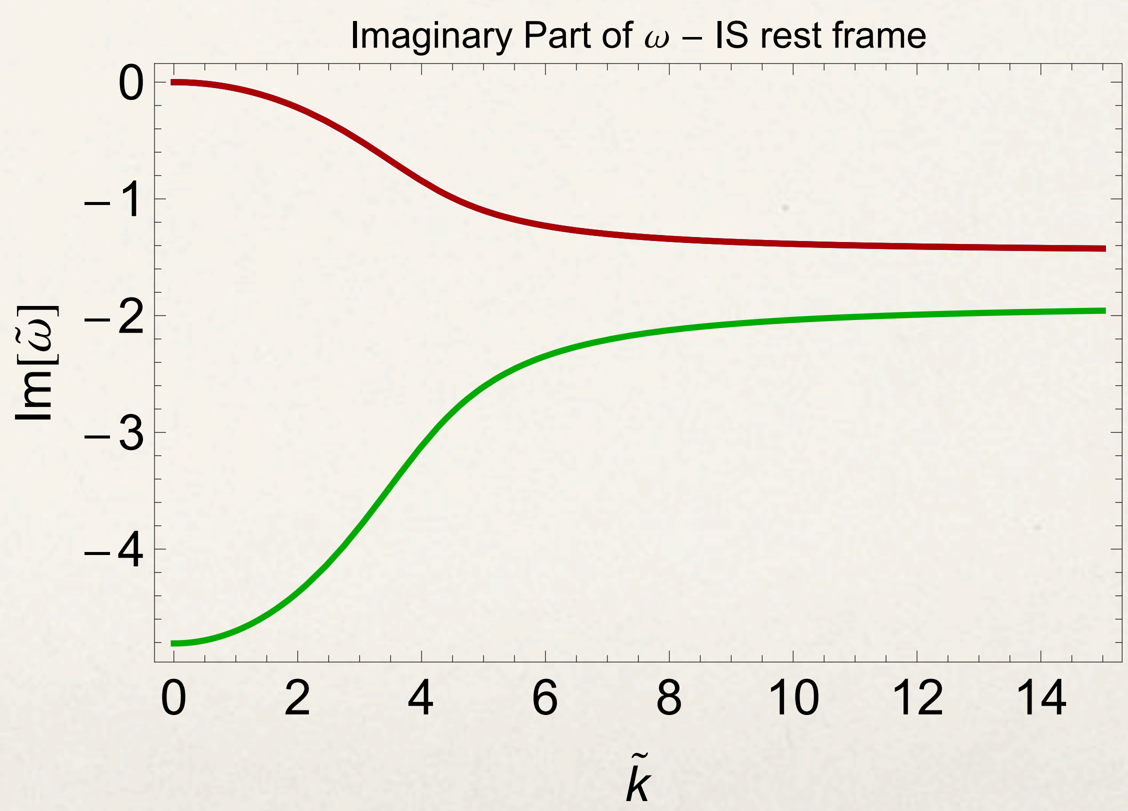

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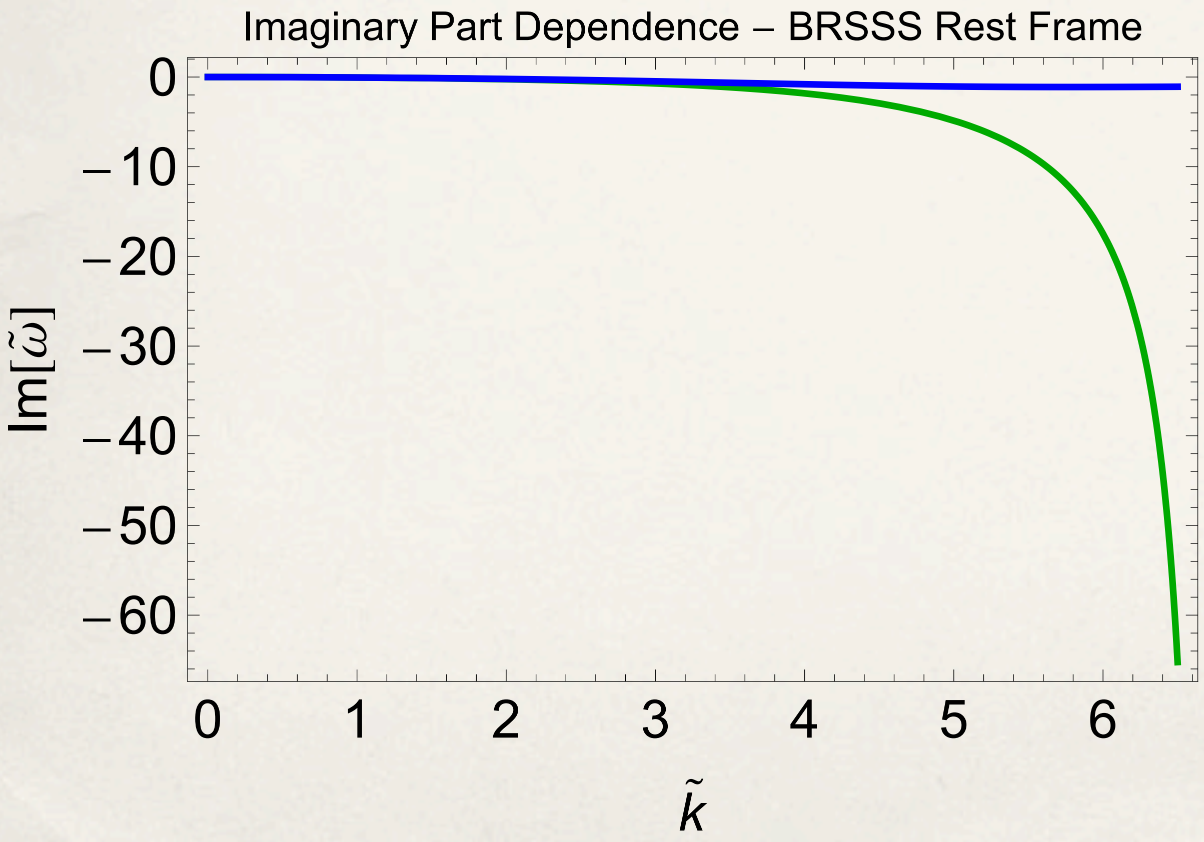

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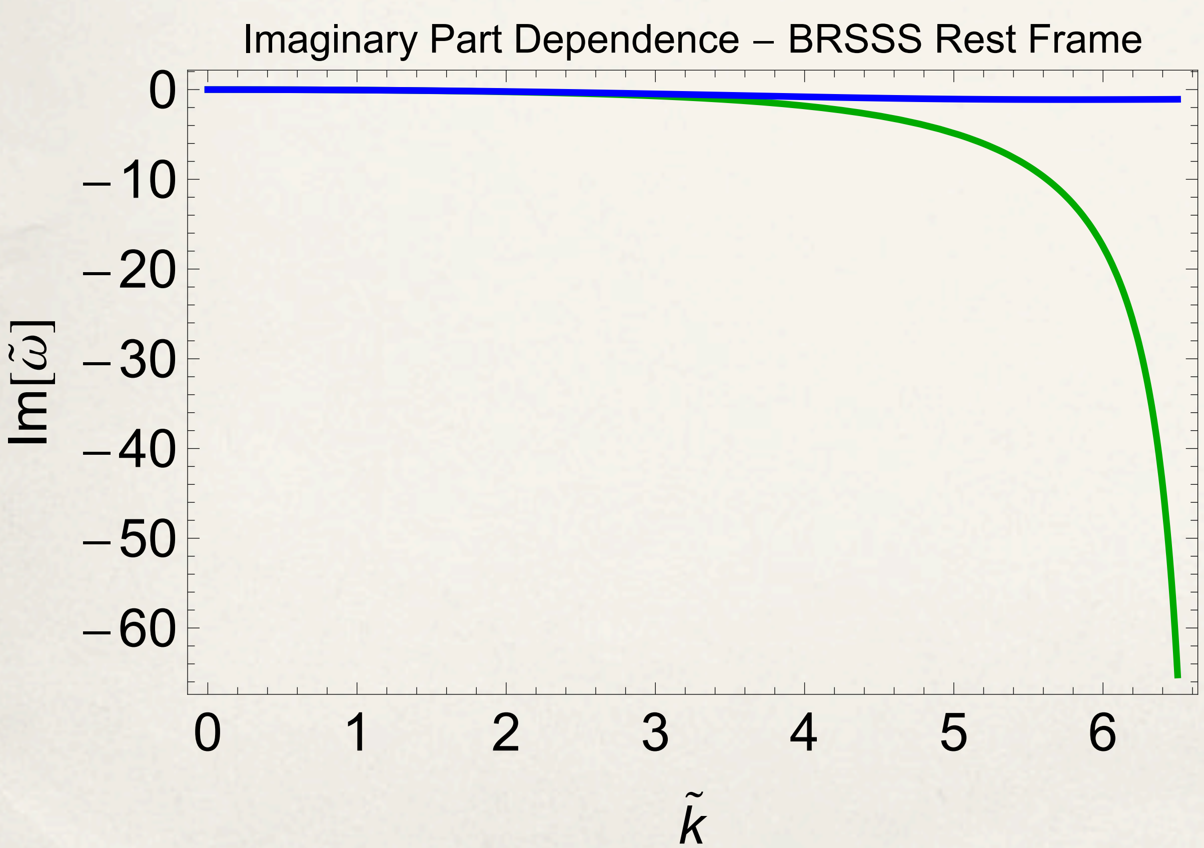

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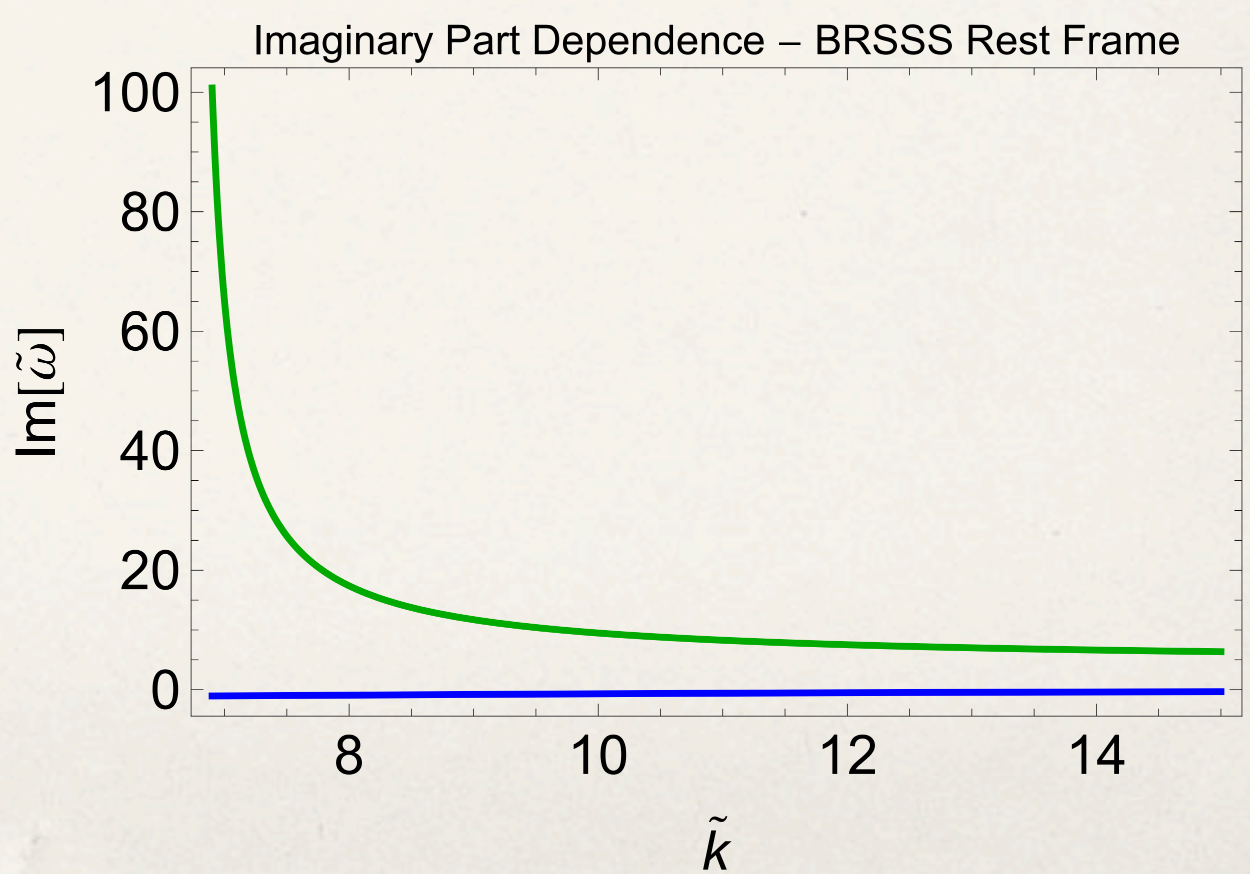

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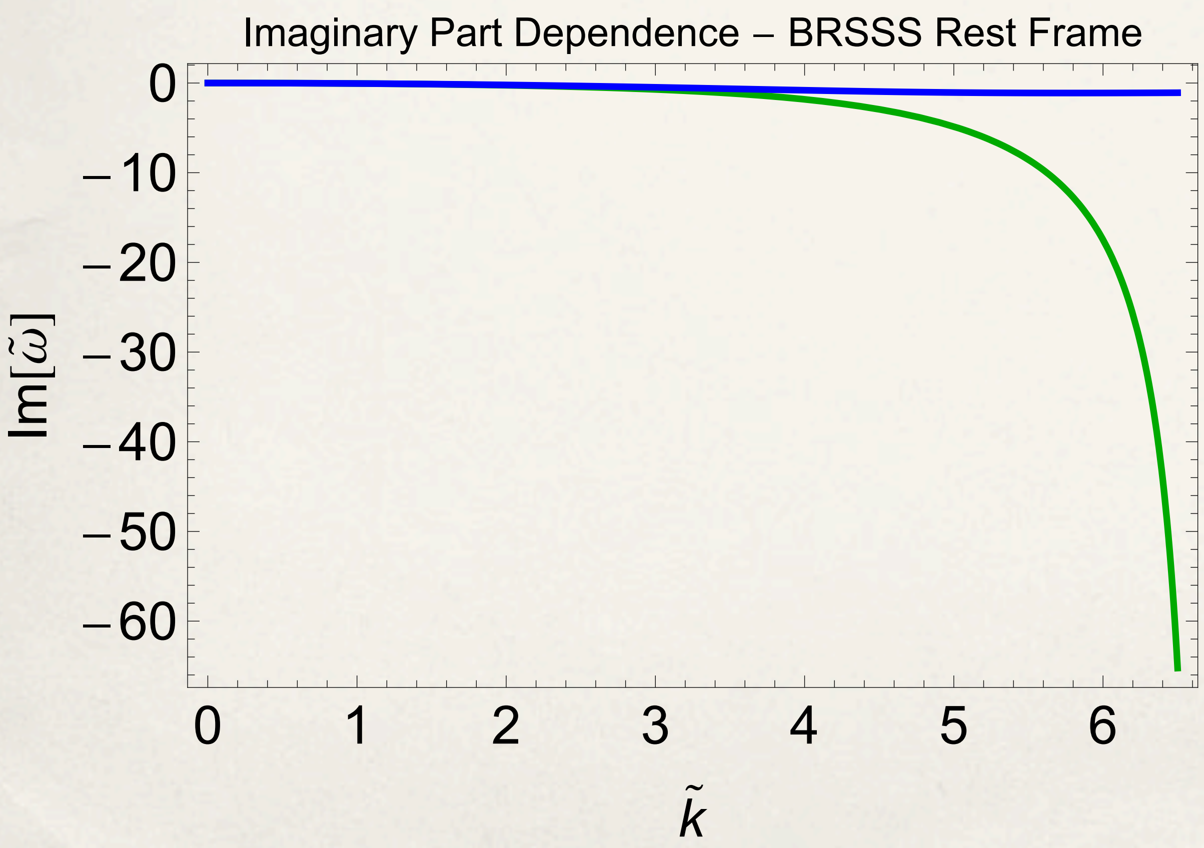

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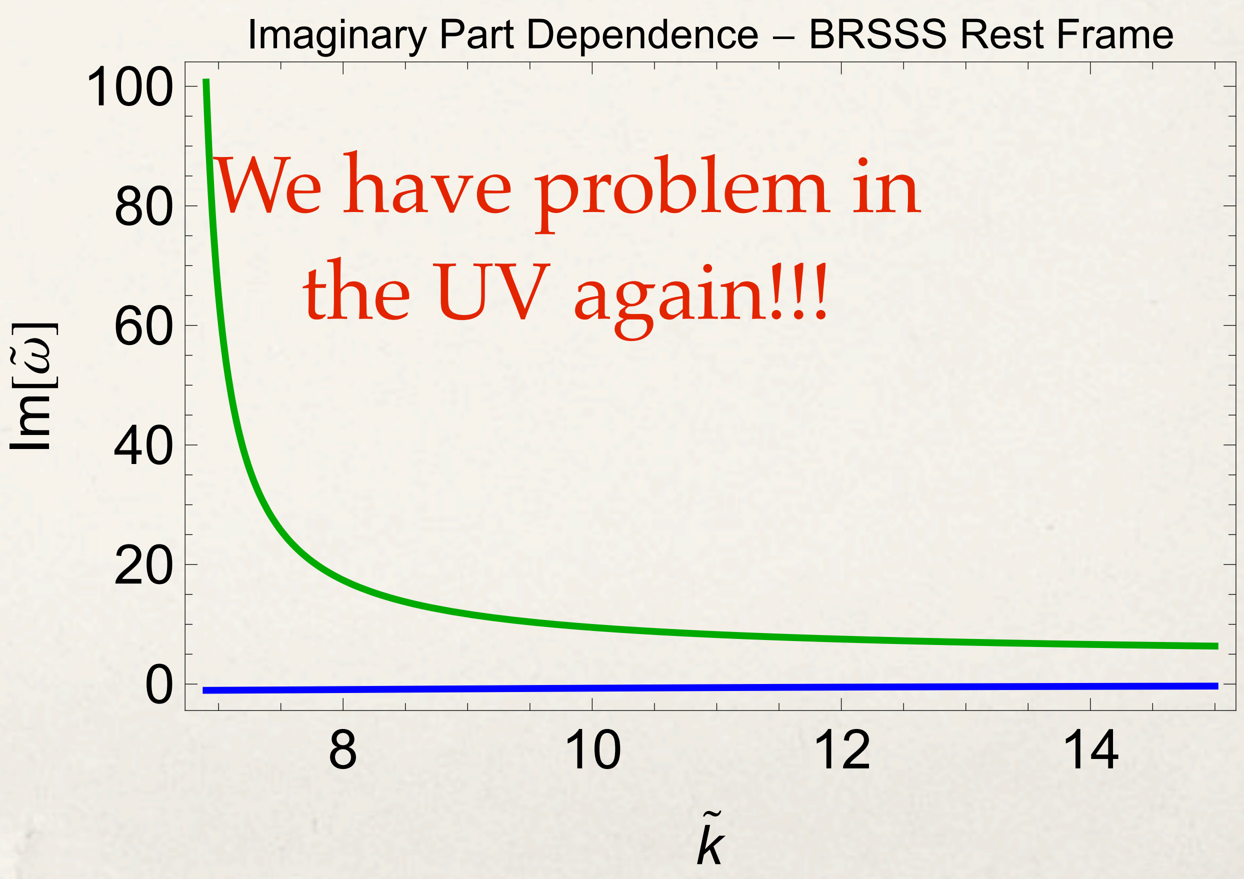

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- IS fine under Lorentz Transformations

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$$u^\mu = (\gamma, \gamma v, 0, 0) + (v\delta u^x(t,x), \delta u^x(t,x), 0, 0)$$

$$T = T_0 + \delta T(t,x)$$

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Necessary for Israel-Stewart

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- We can write the dispersion relation in any frame by this definition!

Dispersion Relations

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$$\tilde{\Omega}^2 - \frac{\tilde{\kappa}^2}{3} - \frac{4\mathrm{i}}{3}\frac{\eta}{s}\tilde{\Omega}\tilde{\kappa}^2 - \frac{4}{3}\frac{\eta}{s}\tilde{\tau}_\pi\tilde{\Omega}^2\tilde{\kappa}^2 = 0$$

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It can go from d=2 to d=4

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Israel-Stewart

Always d=3

$$\tilde{\Omega}^2 - \frac{\tilde{\kappa}^2}{3} - \frac{4\mathrm{i}}{3}\frac{\eta}{s}\tilde{\Omega}\tilde{\kappa}^2 + \mathrm{i}\tilde{\tau}_\pi\tilde{\Omega}^3 + \frac{\mathrm{i}\tilde{\tau}_\pi}{3}\tilde{\Omega}\tilde{\kappa}^2 = 0$$

Causality and Stability (BRSSS)

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$$\tilde{\omega} = \omega_0 + \omega_1 k + \omega_2 k^2 + O(k^3)$$

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BRSSS - 4 Roots

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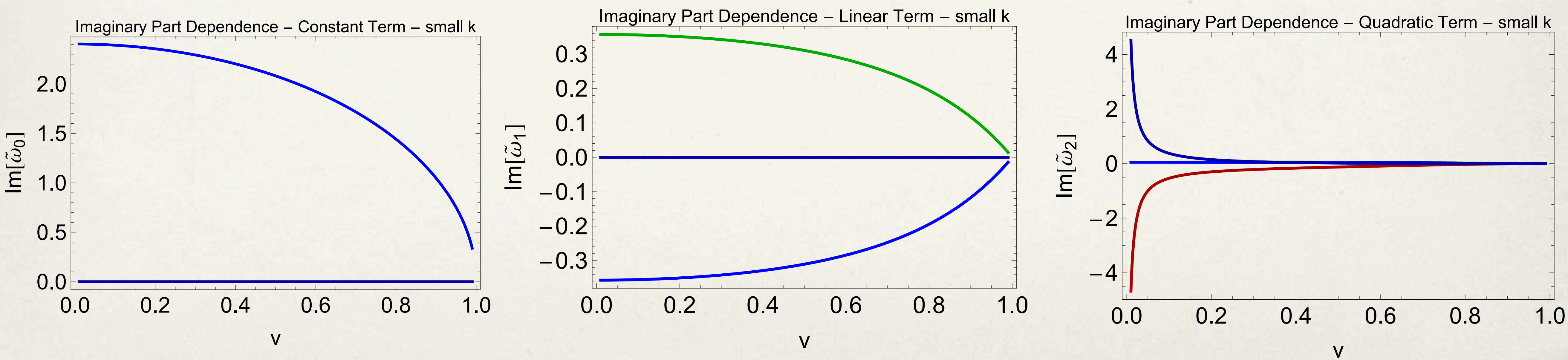
BRSSS - 4 Roots

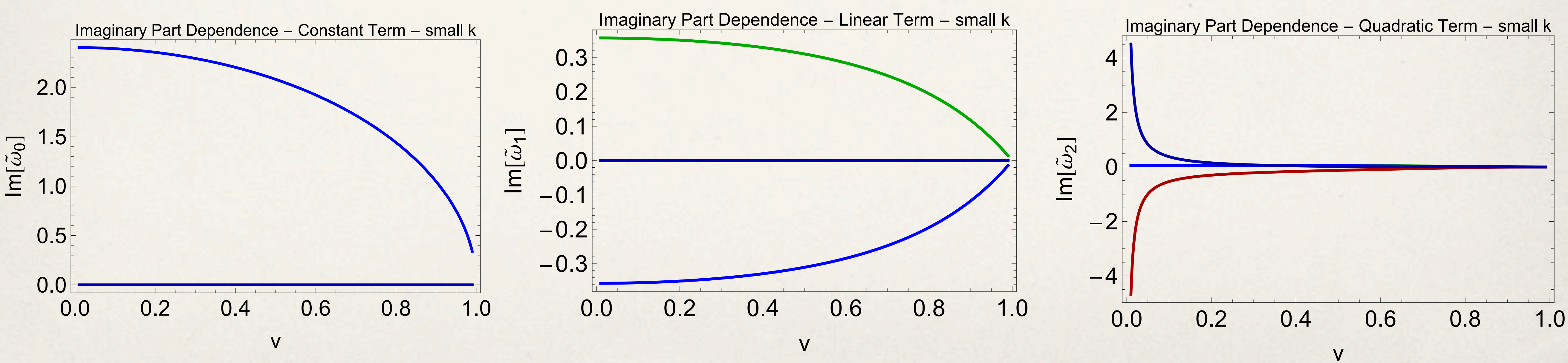


Causality and Stability (BRSSS)

$$\tilde{\omega} = \omega_0 + \omega_1 k + \omega_2 k^2 + O(k^3)$$

BRSSS - 4 Roots

Not Stable even for small v! There are always unstable modes!



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Israel-Stewart - 3 Roots

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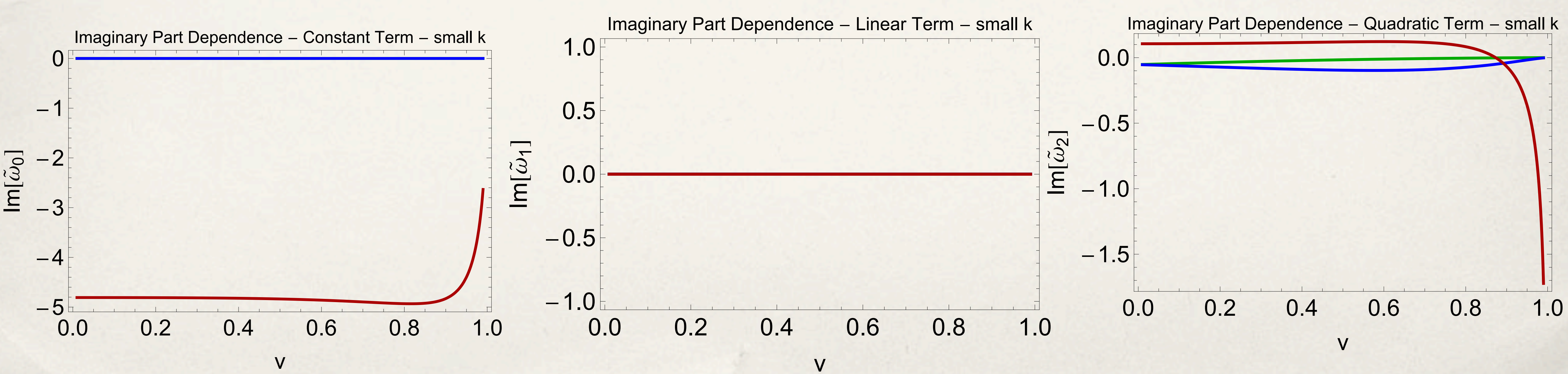
Israel-Stewart - 3 Roots

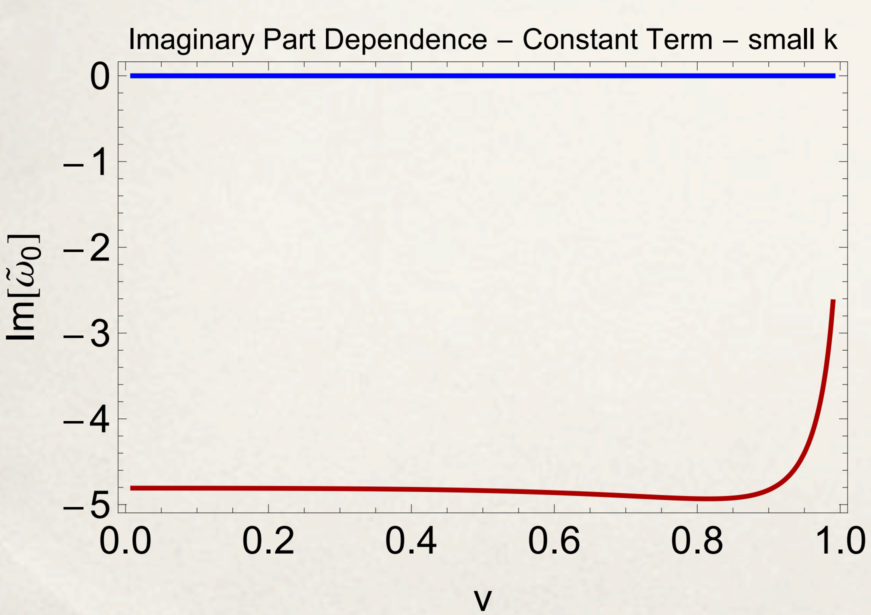


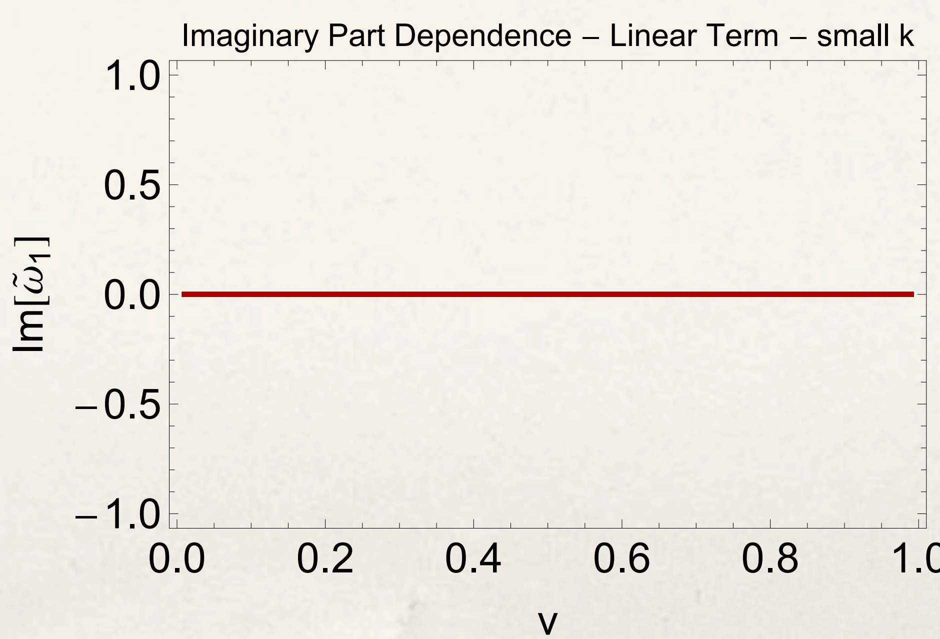
Causality and Stability (IS)

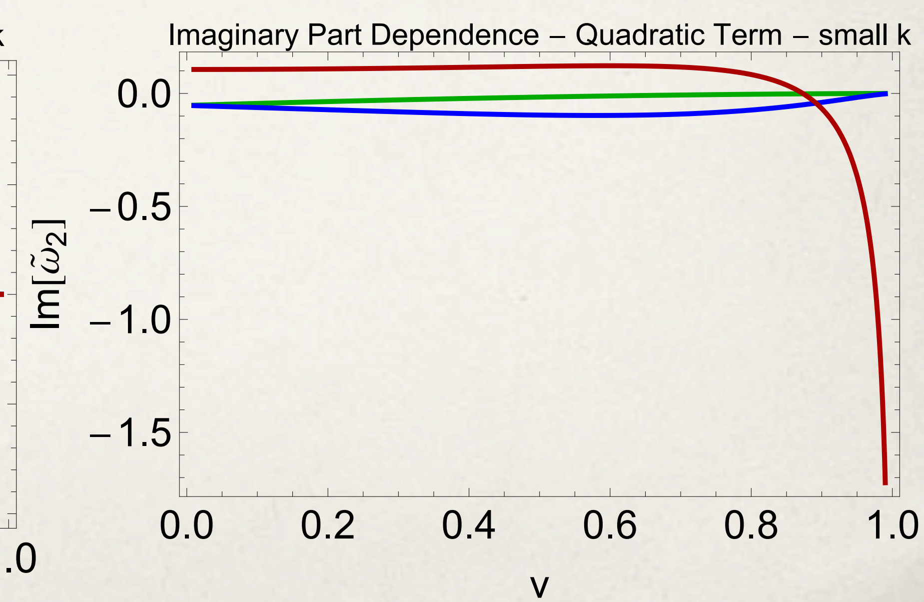
$$\tilde{\omega} = \omega_0 + \omega_1 k + \omega_2 k^2 + O(k^3)$$

Israel-Stewart - 3 Roots

No Problem with Stability!







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- Relaxation is needed to cure causality and stability issues
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- We proved that BRSSS is not stable even for infinitesimal boost in the IR
- BRSSS does not improve Navier-Stokes, it is even less stable!
- BRSSS is not equivalent to Israel-Stewart!!!
- Therefore, τ of BRSSS and τ of Israel-Stewart are different coefficients

Future Work

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- Finish this paper :)

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- Calculate the Israel-Stewart coefficient τ using holography
- Study general transformation properties of Green functions under Lorentz Transformation

Appendix 1 - Overview

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Dissipative contribution Stress
Energy Tensor :

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Appendix 1 - Overview

First, Simplest Relativistic Viscous theory:

Dissipative contribution Stress
Energy Tensor :

$$\pi^{\mu\nu} = T^{\mu\nu} - T^{\mu\nu}_{ideal} = T^{\mu\nu} - \epsilon u^{\mu} u^{\nu} + p\Delta^{\mu\nu}$$

Equations of motion

$$D\epsilon + (\epsilon + p)\theta - \pi^{\mu\nu}\nabla_{\perp(\mu}u_{\nu)} = 0$$

$$(\epsilon + p)Du^{\alpha} - \nabla^{\alpha}_{\perp}p + \Delta^{\alpha}_{\nu}\nabla_{\perp\mu}\pi^{\mu\nu} = 0$$

Appendix 2 - BRSSS and IS

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BRSSS claim their theory is the same as Israel-Stewart

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$$\tilde{\omega}_{BRSSS\,rest,1} = \frac{\tilde{k}}{\sqrt{3}} - \frac{2}{3}\mathrm{i}\frac{\eta}{s}\tilde{k}^2 - \frac{2}{3\sqrt{3}}\frac{\eta}{s}(\frac{\eta}{s} - \tilde{\tau}_\pi)\tilde{k}^3 + O(\tilde{k}^4)$$

$$\tilde{\omega}_{BRSSS\,rest,2} = -\frac{\tilde{k}}{\sqrt{3}} - \frac{2}{3}\mathrm{i}\frac{\eta}{s}\tilde{k}^2 + \frac{2}{3\sqrt{3}}\frac{\eta}{s}(\frac{\eta}{s} - \tilde{\tau}_\pi)\tilde{k}^3 + O(\tilde{k}^4)$$

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BRSSS claim their theory is the same as Israel-Stewart

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$$\tilde{\omega}_{IS\,rest,1} = \frac{\tilde{k}}{\sqrt{3}} - \frac{2}{3}\mathrm{i}\frac{\eta}{s}\tilde{k}^2 - \frac{2}{3\sqrt{3}}\frac{\eta}{s}(\frac{\eta}{s} - \tilde{\tau}_\pi)\tilde{k}^3 + O(\tilde{k}^4)$$

$$\tilde{\omega}_{IS\,rest,2} = -\frac{\tilde{k}}{\sqrt{3}} - \frac{2}{3}\mathrm{i}\frac{\eta}{s}\tilde{k}^2 + \frac{2}{3\sqrt{3}}\frac{\eta}{s}(\frac{\eta}{s} - \tilde{\tau}_\pi)\tilde{k}^3 + O(\tilde{k}^4)$$

Appendix 3 - Dispersion Relations

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In a reference frame parametrized by v:

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$$\tilde{\omega}^4\{\frac{4}{3}\frac{\eta}{s}\tilde{\tau}_\pi\gamma^2 v^2\}+\ \tilde{\omega}^3\{-\frac{4}{3}i\frac{\eta}{s}\gamma v^2-\frac{4}{3}\frac{\eta}{s}\tilde{\tau}_\pi\gamma^2 2v(1+v^2)\tilde{k}\}+$$

$$\tilde{\omega}^2\{-(1-\frac{v^2}{3})+\frac{4}{3}i\frac{\eta}{s}\gamma v(v^2+2)\tilde{k}+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2(v^4+4v^2+1)\tilde{k}^2\}+\tilde{\omega}\{\frac{4}{3}v\tilde{k}-\frac{4}{3}i\gamma\frac{\eta}{s}(1+2v^2)\tilde{k}^2-\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2(2v)(1+v^2)\tilde{k}^3\}+$$

$$\{-(v^2-\frac{1}{3})\tilde{k}^2+\frac{4}{3}i\frac{\eta}{s}\gamma v\tilde{k}^3+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2 v^2\tilde{k}^4\}=0$$

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$$\tilde{\omega}^2\{-(1-\frac{v^2}{3})+\frac{4}{3}i\frac{\eta}{s}\gamma v(v^2+2)\tilde{k}+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2(v^4+4v^2+1)\tilde{k}^2\}+\tilde{\omega}\{\frac{4}{3}v\tilde{k}-\frac{4}{3}i\gamma\frac{\eta}{s}(1+2v^2)\tilde{k}^2-\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2(2v)(1+v^2)\tilde{k}^3\}+$$

$$\{-(v^2-\frac{1}{3})\tilde{k}^2+\frac{4}{3}i\frac{\eta}{s}\gamma v\tilde{k}^3+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2 v^2\tilde{k}^4\}=0$$

Israel-Stewart

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$$\{-(v^2-\frac{1}{3})\tilde{k}^2+\frac{4}{3}i\frac{\eta}{s}\gamma v\tilde{k}^3+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2v^2\tilde{k}^4\}=0$$

Israel-Stewart

$$\tilde{\omega}^3\{-\mathrm{i}\gamma\tilde{\tau}_\pi(1-\frac{v^2}{3})+\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma v^2\}+ \tilde{\omega}^2\{(1-\frac{v^2}{3})+\mathrm{i}\gamma\tilde{\tau}_\pi v\frac{(7-v^2)}{3}\tilde{k}-\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma v(2+v^2)\tilde{k}\}+$$

$$\tilde{\omega}\{-\frac{4}{3}v\tilde{k}+\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma(1+2v^2)\tilde{k}^2+\mathrm{i}\gamma\tilde{\tau}_\pi\frac{(1-7v^2)}{3}\tilde{k}^2\}+ \{(v^2-\frac{1}{3})\tilde{k}^2-\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma v\tilde{k}^3+\mathrm{i}\gamma\tilde{\tau}_\pi v(v^2-\frac{1}{3})\tilde{k}^3\}=0$$

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BRSSS

Degree 4!

$$\tilde{\omega}^4\{\frac{4}{3}\frac{\eta}{s}\tilde{\tau}_\pi\gamma^2v^2\}+\ \tilde{\omega}^3\{-\frac{4}{3}i\frac{\eta}{s}\gamma v^2-\frac{4}{3}\frac{\eta}{s}\tilde{\tau}_\pi\gamma^2 2v(1+v^2)\tilde{k}\}+$$

$$\tilde{\omega}^2\{-(1-\frac{v^2}{3})+\frac{4}{3}i\frac{\eta}{s}\gamma v(v^2+2)\tilde{k}+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2(v^4+4v^2+1)\tilde{k}^2\}+\tilde{\omega}\{\frac{4}{3}v\tilde{k}-\frac{4}{3}i\gamma\frac{\eta}{s}(1+2v^2)\tilde{k}^2-\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2(2v)(1+v^2)\tilde{k}^3\}+$$

$$\{-(v^2-\frac{1}{3})\tilde{k}^2+\frac{4}{3}i\frac{\eta}{s}\gamma v\tilde{k}^3+\frac{4}{3}\tilde{\tau}_\pi\frac{\eta}{s}\gamma^2v^2\tilde{k}^4\}=0$$

Israel-Stewart

Still degree 3

$$\tilde{\omega}^3\{-\mathrm{i}\gamma\tilde{\tau}_\pi(1-\frac{v^2}{3})+\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma v^2\}+\ \tilde{\omega}^2\{(1-\frac{v^2}{3})+\mathrm{i}\gamma\tilde{\tau}_\pi v\frac{(7-v^2)}{3}\tilde{k}-\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma v(2+v^2)\tilde{k}\}+$$

$$\tilde{\omega}\{-\frac{4}{3}v\tilde{k}+\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma(1+2v^2)\tilde{k}^2+\mathrm{i}\gamma\tilde{\tau}_\pi\frac{(1-7v^2)}{3}\tilde{k}^2\}+\ \{(v^2-\frac{1}{3})\tilde{k}^2-\frac{4}{3}\mathrm{i}\frac{\eta}{s}\gamma v\tilde{k}^3+\mathrm{i}\gamma\tilde{\tau}_\pi v(v^2-\frac{1}{3})\tilde{k}^3\}=0$$

Appendix 4 - Causality and Stability (BRSSS)

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The four roots are:

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$$\tilde{\omega}_{1,BRSSS}(\tilde{k},v) = \{\frac{\sqrt{3}}{2}\frac{1}{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}\frac{1}{v} + \frac{\mathrm{i}}{2\tilde{\tau}_\pi} - \frac{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}{4(\sqrt{3}\frac{\eta}{s}\tilde{\tau}_\pi^2)}(\frac{\eta}{s} + 4\tilde{\tau}_\pi)v + O(v^2)\} + \tilde{k}\{\frac{1}{v} - \frac{\mathrm{i}\frac{\eta}{s}}{\sqrt{3}\sqrt{\frac{\eta}{s}\tilde{\tau}_{pi}}} + \frac{v}{3} + O(v^2)\} + O(\tilde{k}^2)$$

$$\tilde{\omega}_{2,BRSSS}(\tilde{k},v) = \{-\frac{\sqrt{3}}{2}\frac{1}{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}\frac{1}{v} + \frac{\mathrm{i}}{2\tilde{\tau}_\pi} + \frac{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}{4(\sqrt{3}\frac{\eta}{s}\tilde{\tau}_\pi^2)}(\frac{\eta}{s} + 4\tilde{\tau}_\pi)v + O(v^2)\} + \tilde{k}\{\frac{1}{v} + \frac{\mathrm{i}\frac{\eta}{s}}{\sqrt{3}\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}} + \frac{v}{3} + O(v^2)\} + O(\tilde{k}^2)$$

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$$\tilde{\omega}_{2,BRSSS}(\tilde{k},v)=\{-\frac{\sqrt{3}}{2}\frac{1}{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}\frac{1}{v}+\frac{\mathrm{i}}{2\tilde{\tau}_\pi}+\frac{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}{4(\sqrt{3}\frac{\eta}{s}\tilde{\tau}_\pi^2)}(\frac{\eta}{s}+4\tilde{\tau}_\pi)v+O(v^2)\}+\tilde{k}\{\frac{1}{v}+\frac{\mathrm{i}\frac{\eta}{s}}{\sqrt{3}\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}+\frac{v}{3}+O(v^2)\}+O(\tilde{k}^2)$$

$$\tilde{\omega}_{3,BRSSS}(\tilde{k},v)=\tilde{k}\{-\frac{1}{\sqrt{3}}+\frac{2}{3}v+O(v^2)\}+\tilde{k}^2\{-\frac{\mathrm{i}}{\sqrt{3}}\frac{1}{v}+(-\mathrm{i}\frac{\eta}{s}-\frac{2}{3}\mathrm{i}(\frac{\eta}{s})^2\tilde{\tau}_\pi)-\frac{\mathrm{i}}{6\sqrt{3}}(-3+4\frac{\eta}{s}\tilde{\tau}_\pi+8(\frac{\eta}{s}\tilde{\tau}_\pi)^2)v+O(v^2)\}+O(\tilde{k}^3)$$

$$\tilde{\omega}_{4,BRSSS}(\tilde{k},v)=\tilde{k}\{\frac{1}{\sqrt{3}}+\frac{2}{3}v+O(v^2)\}+\tilde{k}^2\{\frac{\mathrm{i}}{\sqrt{3}}\frac{1}{v}+(-\mathrm{i}\frac{\eta}{s}-\frac{2}{3}\mathrm{i}(\frac{\eta}{s})^2\tilde{\tau}_\pi)+\frac{\mathrm{i}}{6\sqrt{3}}(-3+4\frac{\eta}{s}\tilde{\tau}_\pi+8(\frac{\eta}{s}\tilde{\tau}_\pi)^2)v+O(v^2)\}+O(\tilde{k}^3)$$

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$$\tilde{\omega}_{2,BRSSS}(\tilde{k},v) = \{-\frac{\sqrt{3}}{2}\frac{1}{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}\frac{1}{v} + \frac{i}{2\tilde{\tau}_\pi} + \frac{\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}}{4(\sqrt{3}\frac{\eta}{s}\tilde{\tau}_\pi^2)}(\frac{\eta}{s} + 4\tilde{\tau}_\pi)v + O(v^2)\} + \tilde{k}\{\frac{1}{v} + \frac{i\frac{\eta}{s}}{\sqrt{3}\sqrt{\frac{\eta}{s}\tilde{\tau}_\pi}} + \frac{v}{3} + O(v^2)\} + O(\tilde{k}^2)$$

$$\tilde{\omega}_{3,BRSSS}(\tilde{k},v) = \tilde{k}\{-\frac{1}{\sqrt{3}} + \frac{2}{3}v + O(v^2)\} + \tilde{k}^2\{-\frac{i}{\sqrt{3}}\frac{1}{v} + (-i\frac{\eta}{s} - \frac{2}{3}i(\frac{\eta}{s})^2\tilde{\tau}_\pi) - \frac{i}{6\sqrt{3}}(-3 + 4\frac{\eta}{s}\tilde{\tau}_\pi + 8(\frac{\eta}{s}\tilde{\tau}_\pi)^2)v + O(v^2)\} + O(\tilde{k}^3)$$

$$\tilde{\omega}_{4,BRSSS}(\tilde{k},v) = \tilde{k}\{\frac{1}{\sqrt{3}} + \frac{2}{3}v + O(v^2)\} + \tilde{k}^2\{\frac{i}{\sqrt{3}}\frac{1}{v} + (-i\frac{\eta}{s} - \frac{2}{3}i(\frac{\eta}{s})^2\tilde{\tau}_\pi) + \frac{i}{6\sqrt{3}}(-3 + 4\frac{\eta}{s}\tilde{\tau}_\pi + 8(\frac{\eta}{s}\tilde{\tau}_\pi)^2)v + O(v^2)\} + O(\tilde{k}^3)$$

Clearly all modes have singularities for arbitrarily small v!

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The three roots are for IS:

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$$\tilde{\omega}_{2,\,IS} = \tilde{k}\{-\frac{1}{\sqrt{3}} + \frac{2v}{3} + O(v^2)\} + \tilde{k}^2\{-\frac{2i}{3}\frac{\eta}{s} - \frac{2i}{\sqrt{3}}\frac{\eta}{s}v + O(v^2)\} + O(\tilde{k}^3)$$

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$$\tilde{\omega}_{3,\,IS} = \{-\frac{\mathrm{i}}{\tilde{\tau}_\pi} - \mathrm{i}\frac{(8\frac{\eta}{s} - 3\tilde{\tau}_\pi)}{6\tilde{\tau}_\pi^2}v^2 + O(v^4)\} + \tilde{k}\,\{(1 - \frac{8\eta}{3s\tilde{\tau}_\pi})v + O(v^3)\} + \tilde{k}^2\{\frac{4\mathrm{i}\eta}{3s} + O(v^2)\} + O(\tilde{k}^3)$$

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All modes are safe for arbitrarily small v!